UNIVERSITY OF LIVERPOOL

From Four Dimensional Instantons to Extremal Black Holes

Thesis submitted in accordance with the requirements of
the University of Liverpool for the degree of Doctor in Philosophy
by

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I would like to dedicate this to my Mum and Dad.
I can't have been the easiest son to be there for.
Through everything and anything.
But I've always known you are there, and always will be.
I love you both.
This is for you.

Declaration

I hereby declare that all work described in this thesis is the result of my own research unless reference to others is given. None of this material has been previously submitted to this or any other university. All work was carried out in the Theoretical Physics Diversion of the Department of mathematical Science, University of Liverpool during the period of October 2006 till September 2010.

This thesis is based upon the work presented in the publications

1. K Waite and T. Mohaupt, Instantons, black holes and harmonic functions, JHEP, 0910:058, 2009.

2. K Waite and T. Mohaupt, Euclidean actions, instantons, solitons and supersymmetry, Journal of Physics A, 2011 (accepted for publication)

Abstract

Instanton solutions of four-dimensional Euclidean sigma models with commuting shift symmetries are constructed within this thesis and their properties evaluated. They are shown to be expressible in terms of harmonic functions when they take values in a completely isotropic submanifold of the target space. Whilst this condition is shown to be a natural product of imposing a Euclidean BPS condition, these solutions are generated without requiring supersymmetry. Instead, they require an integrability condition to be met, which allows supersymmetric and non-supersymmetric multi-centered solutions. This large class of solutions includes supersymmetric instanton solutions of $\mathcal{N} = 2$ vector multiplets as a sub-class. The existence of real solutions requires that the target space carries a metric of indefinite signature. The integrability condition further requires it to be a para-Kähler manifold which is obtained from a real Hessian manifold by a generalised r-map. These instanton solutions exist despite Derrick's Theorem and this is shown from three separate perspectives: the standard Wick rotated theory, where instantons correspond to complex saddle points, a modified Wick rotated version where all axionic scalars are analytically continued, so that the saddle points become real at the expense of an indefinite Euclidean action, and the scalar-tensor theory where all axionic scalars have been dualized into tensor fields. In addition to finding instanton solutions we also show how they contribute to transition amplitudes in a consistent saddle point approximation. With the proper choice of integration contours and boundary conditions the different perspectives are shown to be related by change of variables. By coupling the sigma model to gravity and lifting to five-dimensional space-time, the instanton solutions are used to construct extremal multi-centred black hole solutions. The properties of these black hole solutions are investigated and they are shown to exhibit particular fixed point behaviour. The attractor equations are shown to take the same form as in supersymmetric theories, however they do not rely upon Killing spinors, or similar approaches but follow directly from second order scalar field equations.

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Acknowledgements

Where do I start? All journeys have beginnings, middles and ends, and all encounter difficulties along the way. I would be lying if I said this was a journey of which I had enjoyed every minute. However, I would also be wrong to say it hasn't been a rewarding one as well. Even if many of those rewards were adjacent to the PhD. For me, this PhD has been a lot like climbing a mountain on a day with low cloud... only now I am stood at the summit with the clouds thinning to leave a view I never thought I'd get!

Throughout this journey there have always been people there helping me, and guiding me through it. However I would like to thank my supervisor, Thomas, who has been a fantastic source of support along the way. When it came to supervisors I really couldn't have asked for more. From the academic support through to the emotional understanding, and to the tireless effort he has been willing to put in to help me through these four years. Looking back, I can only wish I could have lived up to my potential and been more the student his supervision deserved! However, despite all my problems, and despite being what I am sure was a continual disappointment, he never gave up on me. And here I am... handing in something that at many stages I never thought I would come close to producing. Thank you Thomas. I know its cliché, but this thesis could never have happened without your continual support.

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Which moves me onto a tougher one for me to write about. The person who actually saved my PhD at the start of my second year, and despite everything that's happened has been central to my time and personal development at Liverpool. Firstly, I just want to that I'll stick to the positives here and ignore the rest that could have been so different. So Gary, I just want to say thank you. In the long run, the impact you've had on me, and my personal journey over the PhD has been huge, regardless of everything else. Just stop and compare me now to the Kirk you first met. You saved my PhD and made me a better person. A much better person. That wouldn't have happened without you there... especially in that first year. For that alone, I will always be grateful and happy we could be friends.

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However, this is my PhD thesis, not a walking thesis, and hence back to my work in the department. Which of course I would never have been able to complete or enjoy without some great office friends for company. Rob, Ben, Frank, Chris L, Dave (+Amelia) all added up to a great, if slightly distracting office! At least I can now juggle! There was also the string group, from which I thank: Alon, Thomas, Radu, Owen, Ben, Elisa, Viraf, Will, Chi, Kyriakos, Miriam and Christina. However I also felt it was important to be a part of things beyond the work as well. So I also want to thank those involved in the squash ladder (Tim and Steve who haven't been mentioned yet) and the regular football which kept me fit and looking forward to Thursdays!

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However, this journey also began long before Liverpool so I just want to mention a few of the other names that have somehow got me from the worst primary school in the country through to handing in my PhD thesis: Mr Theoff - for the love of science he installed in me; Mr Sewell - my DGSB form tutor; Mr Callacher - my maths teacher; Craig - my best friend when I was younger, Daniel, Lewis and Paul - who have stayed friends and supported me throughout and finally all the Durham Grey College people (too many to name!) for also being there no matter what happens! Thank you all, so much for helping me achieve this. It means a lot to me to finish this!

And there are many more people I could have mentioned. The psychology girls (including Aaron), friends from home, people i played poker with, Paul and Kaj whom I lived with, and the many more people I have enjoyed having the chance to meet while in Liverpool... however I really have to stop at some point here!

Before I do though, I want to give a final big thank you to my family. To my mum, my dad, my brilliant 2 sisters Katie and Kelly, my grandparents and everyone else who has supported me along the way. I love you guys and thank you for everything you have done for me.

So whilst I am now off to forge a new career, getting a PhD was an ambition I had from when I was very young. Ok, its not in Paleontology... but its close(ish)! However, whilst I still have one final hurdle to cross to achieve that goal, I wouldn't have come anywhere near close without any of you guys there.

So thank you to all of you,

Kirk

Chapter 1

Introduction

1.1 Instantons

The main content of this thesis is to investigate the concept of an instanton solution, when such solutions exist and how they behave under dimensional lifting. We will therefore start this section by discussing the concept of an instanton solution through reviewing the chapter and treatment by Coleman in his Erice lectures [1] and the book by Rajaraman [2]. The discussion will motivate the instanton from a quantum mechanical perspective as a semiclassical solution to the problem of quantum tunnelling. We will then briefly discuss the roles that instantons have in quantum field theories, before returning our focus towards instantons from a purely scalar field theory and a discussion of a potential problem we have to overcome.

1.1.1 An alternative to the WKB Approximation

The calculation of physical observables in a large proportion of quantum field theories often requires the use of perturbation theory. In perturbation theory the coupling constant is assumed to be small, and hence can be used as the coefficient of expansion. However, there are limitations to the applicability of such expansions and this can be seen with the basic quantum mechanical problem of the tunneling amplitude of a particle through a potential barrier. If we consider a particle in a one dimensional potential well $V(x)$

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 - V(x) \tag{1.1}$$

we obtain the Schrödinger equation

$$\frac{d^2\psi}{dx^2} = \frac{2}{\hbar^2}(V(x) - E)\psi. \tag{1.2}$$

By then assuming that $V(x)$ varies slowly in comparison to the solution, we can rewrite this equation as a first order differential equation

$$\frac{d\psi}{dx} = \pm\frac{\sqrt{2}}{\hbar}(V(x) - E)\psi. \tag{1.3}$$

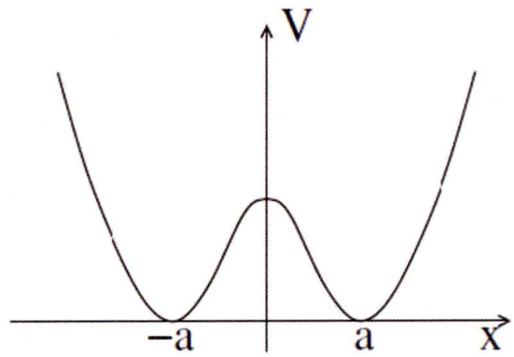

Figure 1.1: Double Well potential $V(x)$

This can be seen by differentiating and then dropping the V' term. It then follows that the amplitude for a particle to tunnel from point a to point b is given by

$$A \sim \exp\left(-\frac{1}{\hbar}\int_a^b \sqrt{2\left(V(x)-E\right)}dx\right), \tag{1.4}$$

which is known as the WKB formula. This is a semi-classical approximation, in that it requires us to assume $\hbar$ is small in order for us to rewrite our Schrödinger equation. However, we can obtain the same result from an alternative approach, by considering the transmission of the particle through the barrier of a double well potential, such as that shown in figure 1.1. In order to calculate this we consider the Feynman path integral in Euclidean space. The formal definition of the amplitude we would like to calculate with the path integral is

$$\langle x_f|e^{-HT/\hbar}|x_i\rangle = N\int [dx]e^{-S/\hbar}. \tag{1.5}$$

where S is the action for our single particle in the potential well $V(x)$.

$$S_E = \int_{\tau_a}^{\tau_b}\left(\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right)d\tau \tag{1.6}$$

This differs from the Minkowski definition of such a path integral as we have Wick rotated our model, by analytically continuing the theory to Euclidean space-time ($t \to i\tau$) [3, 4]. This Wick rotation is performed in order to ensure that the path integral is definite, and hence of a form that we can evaluate. By having the equivalent effect of rotating the potential by 180° as shown in figure 1.2 the Euclidean path integral is then convergent.

One aspect of this thesis will involve a discussion of how the form of the action effects the solutions that we can generate. However for our immediate discussion it is sufficient

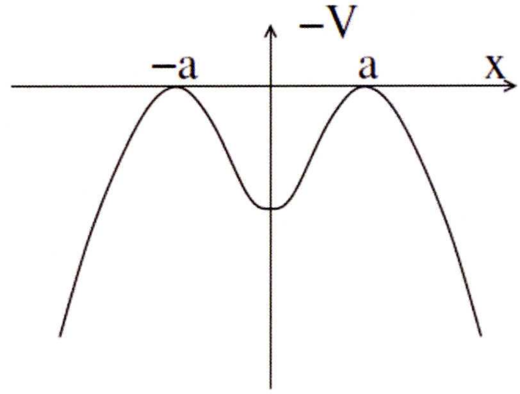

Figure 1.2: The Wick rotated, and hence inverted potential $V(x)$

to know that we can use the properties of such a path integral in the semi-classical limit to evaluate a solution to the tunneling problem. By applying the semi classical limit, we expect that the functional integral to be dominated by the stationary point that minimises the action S. Hence, if we make the assumption of a single stationary point we can see that

$$-\frac{d^2x}{d\tau^2}+\frac{dV}{dx}=0 \tag{1.7}$$

This is identical to the classical equation of motion for a particle in a potential $-V(x)$ and solving this equation we see

$$\frac{dx}{d\tau}=\sqrt{2V} \tag{1.8}$$

and hence the action can be rewritten as

$$S_0=\int_{\tau_a}^{\tau_b}\sqrt{2V}dx \tag{1.9}$$

which is identical in form to the WKB approximation.

We can plot the solution to this as a classical trajectory, called a kink solution and shown in figure 1.1.1. This trajectory has values $\pm a$ as Euclidean time $\tau\to\pm\infty$ and connects these two regimes through a well localised region. This is the instanton solution (or anti-instanton if it is connecting $\mp a$ as $\tau\to\pm\infty$) and they have a structure that has very similar properties to solitonic solutions which appear in classical field theory and which we will discuss in more detail in the next section.

We can see that for large τ, the instanton approaches a and hence we can approximate the equation of motion as

$$\frac{dx}{d\tau}=\omega(a-x). \tag{1.10}$$

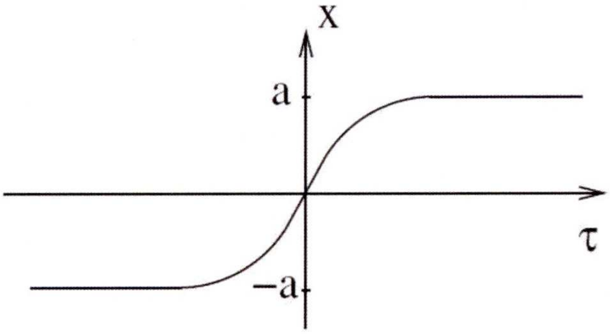

Figure 1.3: Kink solution from $-a$ to $+a$

It follows that for large τ

$$(a - x) \propto e^{-\omega\tau}. \tag{1.11}$$

Thus the instanton solution is a localised one with a size of order $1/\omega$.

1.1.2 Instantons in Quantum Field Theories

The instanton of quantum tunneling provides a simple example of a calculation within quantum theories that relies upon non-perturbative concepts to evaluate the observable physics from those theories. However, instanton calculus is a much more powerful subject which is not restricted to this simple example.

Instantons appear in a number of different theories and have important roles in many modern concepts within theoretical physics. These range from the D-Instanton of Type IIb [5] string theory to instantons within non-abelian gauge theories such as Yang Mills [6]. These instantons are important structures as they have potential effects upon calculations such as the transition amplitudes by providing additional leading terms which we have already seen cannot be observed through perturbation theory alone. However, the standard mathematics textbook definition of an instanton is generally too narrow to encompass the variety of solutions observed (and equivalently soliton solutions which we will discuss in the next section).

We will therefore be using a more relaxed definition, where for our purposes an instanton can be defined as a solution to the Euclidean equations of motion which have finite, non-zero action. A particular difference is that the standard definition also requires that they are regular solutions, however for our purposes we will also include solutions with finite sources. Consequently our definition of an instanton corresponds to those seen within the current physics literature rather than those within mathematics textbooks.

Within this thesis we are only going to be considering abelian theories which can be connected to supergravity, and hence our focus will be on instantons which are very similar to the D-instanton. Our starting place will therefore be non-linear sigma models of the form

$$S = \int d^D x G_{ab}(\phi)\partial_\mu \phi^a \partial^\mu \phi^b \tag{1.12}$$

without a scalar potential. This action is general enough to cover a variety of interesting cases, including the instanton solutions of Type IIb string theory and the vector multiplets of four dimensional $N = 2$ models. Within these models we would like to investigate the existence of instanton solutions, their properties and how they then lift to higher dimensional theories. Within section 2, we discuss the construction of these solutions from a four dimensional purely scalar action, and the form and behaviour of these solutions.

1.1.3 Derrick's Theorem

Currently we have only considered the details of a one dimensional problem within the framework of quantum mechanics, however the models for which we are going to be interested in constructing instanton solutions are four-dimensional scalar sigma models. Consequently it would be informative to investigate an example of higher dimensional solutions within this framework. Unfortunately at this point we run into a discouraging no-go theorem with Derrick's Theorem. This forbids the existence of instantonic (or solitonic) solutions for space-like dimensions of $D > 2$ for purely scalar theories with a positive definite metric, and only allows for the existence at $D = 2$ in models without a scalar potential. [1, 7]

To see why this is the case consider a set of scalar fields $\phi^a(x)$ which are solutions to the Euler-Lagrange equations of a Euclidean action of the form (1.12) with an additional potential field. We can now define a one-parameter family of field configurations by

$$\phi^a(x|\lambda) := \phi^a(\lambda x) \tag{1.13}$$

with positive parameter λ.

The kinetic term of our Lagrangian then takes the form $\lambda^{(2-D)}V_1$, where V_1 denotes the kinetic terms, whilst any potential term would similarly have the form $\lambda^{-D}V_2$, with V_2 denoting the potential terms. Hence the we can see that our action is given by

$$V(\lambda) = \lambda^{(2-D)}V_1 + \lambda^{-D}V_2 \tag{1.14}$$

From our original definition we know that $\lambda = 1$ is a solution. Thus $\lambda = 1$ must be a critical point of the action. By evaluating the action on this solution for a positive definite metric, this then implies that either ϕ is a trivial constant field or that we can solve

$$(D-2)V_1 - DV_2 = 0. \tag{1.15}$$

For $D > 2$ we see that V_1 and V_2 must vanish, and hence the only viable solution is that ϕ^a is a constant ground state of our original theory and no higher order solutions exist. Thus we have no interesting terms that would contribute to any sub-leading terms of any quantum amplitude calculation. Consequently the only solutions possible in this case are the trivial ground states. Furthermore, for $D = 2$, we can see that this equation only requires the vanishing of V_2, and hence we can expect instanton solution only to exist with the absence of a scalar potential. Hence the only possible non trivial solutions we have for such models occur only when $D = 1$.

It is also worthing noting that this differs slightly from the argument for soliton solutions as within this case it is possible to use the Hamiltonian to show that no non-trivial solitonic solutions are possible for $D = 2$ regardless of the existence of a potential. The key change is that you can now use Hamiltons principle which states that the energy is stationary on the solution, and hence

$$(D-2)V_1 + DV_2 = 0. \tag{1.16}$$

Thus for this equation non-trivial solutions only exist for $D < 2$. However $D = 2$ is also constrained by reapplying Hamilton's principle to V_1 alone as the above equation only constrains V_1 to vanish in this case. Hence, similar to the instanton non-trivial solutions for $D > 1$ are prohibited by Derrick's Theorem.

Evidently this is a crucial problem to our desire to investigate instanton solutions in higher dimensional theories. However, it has already been mentioned that these instanton solutions do exist within a large number of current models and are used throughout theoretical physics. These inclde the axionic wormhole-type solutions [8, 9, 10], D-Instanton solutions of type IIb supergravity [5, 11] and the hypermultiplet and vector multiplet instanton solutions in $\mathcal{N} = 2$ compactifications [12, 13, 14, 15, 16, 17]. We would therefore like to address this apparent contradiction and answer the question of how is it possible to construct such solutions despite these apparent restrictions provided by Derrick's Theorem. This forms the main focus of this thesis which explains and justifies the usage of such instanton solutions through the analysis of a general class of models. The crux of the issue lies within the assumptions that are made when the argument for Derrick's Theorem is presented. By investigating and relaxing these assumptions this thesis successfully provides a number of different approaches to avoid these restrictions.

The first of these assumptions we can consider is the form of the saddle point we expect to see within the theory. In quantum mechanics and quantum field theories, we are not restricted to only looking at real saddle points. Instead we can have integrals over real contours that are dominated by complex saddle points, and hence our standard Euclidean action can be kept but the solutions can now be considered to be finite non-constant and complex instead.

Another assumption we can look at is the form of the target metric of our model. In the above argument we have assumed a positive definite metric, such that the functional integral remains damped. However, as we will discuss later in this thesis, it is possible to formulate the theory with either an indefinite or definite metric. We can then impose appropriate boundary conditions such that we can evaluate indefinite functional integrals. This then requires an appropriate choice of integration contour in the complexified field space, such that we still have a convergent functional integral.

Symmetries within the theory can also sometimes be used to construct the theory in a dual form. In the case of the sigma models that we are interested in, the shift symmetries of some of the scalar fields. We will refer to these as axionic scalars, and hence these axionic shift symmetries allow us to reformulate the model in terms of dualized antisymmetric tensor fields. This is then a gauge theory, to which Derrick's Theorem no longer applies and instanton solutions can be obtained from a positive definite action which are equivalent to the scalar forms.

All of these assumptions are discussed at length with respect to the sigma model that we are interested in, and reviewed in chapter 3 when we discuss the properties of the instanton solutions that we generate.

1.2 Solitons and Black Holes

1.2.1 Solitons in Quantum Field Theories

Once we have constructed instanton solutions and investigated their properties, we would then like to lift these solutions to higher dimensional solutions. Without including gravity within our theories, these dimensionally lifted instantons can be interpreted as charged soliton solutions of an appropriate theory of gauge fields and matter. Solitons are very similar to the instanton solutions that we have discussed previously, in that they are regular, non-dissipative solutions of finite energy. Consequently they include, but are not limited to, the kink solutions of two dimensional quantum field theories, D-branes of string theories and monopoles from Yang Mills and Yang Mills-Higgs theories [1]. However, similarly to the instantons, we will also refer to objects that include hidden singularities as solitons. Hence our definition of a solution include one of the important gravitational solutions within the context of modern theoretical physics. These are the classical black hole solutions of Einstein gravity.

Within chapter 5, we shall look at the form of our instanton solutions when we lift them to higher dimensional objects whilst including gravity within the framework. This will involving taking our four dimensional Euclidean theory (4,0) and solutions and extending them to a five dimensional theory with time (4,1). These solutions will then be shown to lift to static black hole solutions which have a variety of properties which we will then investigate. [18, 19] This introductory section will therefore present the expected form of such black hole solutions, and discuss what properties are associated with these solutions.

1.2.2 The Reissner-Nordström Black Hole

The starting point for our discussion of black holes is with the Einstein Hilbert action, which describes gravity as a classical theory

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R. \tag{1.17}$$

The most general stationary black hole solution to this action is the Kerr-Newmann black hole, however we are only interested in a static and hence stationary subclass of these solutions. A black hole is stationary if we can find a time independent coordinate system to describe it and is much less restrictive than static. For a black hole to be static we require that it possesses a Killing vector which is orthogonal to a family of space-like surfaces for constant time. If these surfaces coincide with the black hole horizon, this is called a Killing horizon. We do not require a detailed analysis of these concepts, however it does allow us to restrict our interest within this thesis to the most general static and stationary solution which is the Reissner-Nordström black hole.

Following the arguments given in D'Invernos textbook [20] and the review by Mohaupt [21], we can therefore construct the solution for such a static charged black hole, by first considering spherically symmetric solutions of the Einstein-Maxwell field equations. We use Plankian units, such that $\hbar = c = 1$, and also set $G_M = 1$. We start with the field equations in a form given by the Ricci Tensor

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}. \tag{1.18}$$

whilst we also require that in source-free regions the Maxwell tensor F_{ab} satisfies the Maxwell equations. For the content of this thesis the trace-reversed equation of motion can then be ignored as we are interested in vacuum solutions of the Reissner-Nordström metric such that the trace of the energy momentum tensor T^{ab} vanishes. It follows that we have the reduced field equation

$$R_{ab} = 8\pi T_{ab}. \tag{1.19}$$

We are looking for spherically symmetric solutions, and hence we can write the canonical form of the line element as

$$ds^2 = -e^{-2h(r)} dt^2 + e^{2k(r)} dr^2 + r^2 d\Omega^2 \tag{1.20}$$

where $d\Omega^2 = \sin^2\theta d\phi^2 + d\theta^2$.

By assuming that the charge is centred at the origin of our coordinates, and hence that the Maxwell tensor is singular at this point with a standard electrostatic type potential, then our functions describing the line element depend only upon the radius. By using these assumptions, along with the asymptotic behaviour of the electrostatic potential we can then obtain the Reissner-Nordström solution

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2 \tag{1.21}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{1.22}$$

and M and Q are identified as the mass and charge of the black hole respectively. Alternatively we could also have a magnetically charged black hole, however we would have to replace the electric charge, Q^2 with a mixed magnetic and electric charge $Q^2 + P^2$. We will concentrate on the electrically charged solution as this will be the most applicable to our dimensionally lifted instanton solutions. We could also extend this solution to a multi-centered one by choosing a more general harmonic function for $f(r)$. In this case we would have

$$f(r) = 1 - \frac{2M}{|\vec{x} - \vec{x_i}|} + \sum_{i=1}^{N} \frac{Q_i^2}{|\vec{x} - \vec{x_i}|^2}. \tag{1.23}$$

where N is the number of centres that we have. We expect to obtain such multi-centred solutions, and a large part of this thesis will be based upon motivating their existence independently of supersymmetry. However, we will continue our analysis for just the single centred case as the properties we obtain are simple to generalise to the multiple case.[22]

The Reissner-Nordström solution can be split into three separate cases that are defined according to their horizon behaviour. In the case $M > |Q|$, we have a non-extremal Reissner-Nordström black hole and we observe two horizons: An exterior event horizon and an internal horizon, referred to as the Cauchy horizon. This is the general non-extremal solution. Alternatively, in the case $M < |Q|$, we have an unphysical solution with a naked singularity. By imposing the absence of such naked singularities this therefore provides a mass bound for space-time as discussed by Gibbons and Hull in [23]. However, When this bound is saturated, we have $M = Q$, and the two horizons coincide, we can determine the area this event horizon as

$$A = 4\pi Q^2 \tag{1.24}$$

This is the extremal limit, and is the case that is most relevant to this thesis and hence the one that we shall consider in detail. The non-extremal case is considered in the further work by Mohaupt and Vaughan [24]. It is also worth noting that the special case of the Schwarzchild black hole is obtained by setting $M > 0$, $Q = 0$.

Currently we have only considered black holes in four dimensions, however the relevant focus of this work is the lifting from four dimensional instantons to five dimensional black holes. Myers and Perry showed in [25] that the generalisation to higher (n+1) dimensions is given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{n-1}^2 \tag{1.25}$$

where $d\Omega_{n-1}$ is the line element on the (n-1)-sphere. The function $f(r)$ is given by

$$f = 1 - \frac{2M}{r^{n-2}} + \frac{Q^2}{r^{2(n-2)}} \tag{1.26}$$

Hence our five dimensional line element is given by

$$ds^2 = -\left(1 - \frac{2M}{r^2} + \frac{Q^2}{r^4}\right)dt^2 + \left(1 - \frac{2M}{r^2} + \frac{Q^2}{r^4}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \tag{1.27}$$

It therefore follows that the extremal five dimensional black hole has the same bound as the four dimensional case $|Q| = M$.

1.2.3 Properties of Black Holes

Black holes as physical objects have a variety of physical properties which can be investigated. We have already discussed the charge of a black hole and the behaviour this entails, however it is also worthwhile to provide a definition for the mass. An intuitive way to measure the mass of a black hole would involve studying the motion of a test particle within the asymptotic region of the space-time where the particle would be expected to see a standard Newtonian potential. Equivalently as our space-time approaches flat space at infinity we can also define the ADM mass of the black hole [17]. Given a general line element of $(n+1)$ space-time

$$ds^2 = -h_{tt}dt^2 + 2h_{t\mu}dtdx^\mu + h_{\mu\nu}dx^{\mu\nu} \tag{1.28}$$

then using the fact that the spatial part $h_{\mu\nu}$ approaches a flat Euclidean metric at infinity

$$ds^2_{flat}\big|_{r\to\infty} = dr^2 + r^2 d\Omega_{n-1}^2 \tag{1.29}$$

we can choose a spatial hypersurface with the spherical asymptotic boundary S_∞^{n-1} and define the ADM Mass as

$$16\pi M_{ADM} = \oint_{S_\infty^{n-1}} d\Sigma^\mu \left(\partial^\nu h_{\mu\nu} - \partial_\mu(\delta^{\rho\sigma}h_{\rho\sigma})\right) \tag{1.30}$$

Using the expansion of the metric for the Reissner-Nordström solution

$$g_{rr} = \left(1 - \frac{2M}{r^2} + \frac{Q^2}{r^4}\right)^{-1} = 1 + \frac{2M}{r^2} + \dots \tag{1.31}$$

we can write

$$h_{\mu\nu}dx^{\mu}dx^{\nu} = \left(1 + \frac{2M}{r^{n-2}} + \dots\right)(dr^2 + r^2 d\Omega^2_{n-1}) \tag{1.32}$$

we can then evaluate this integral and obtain the expression for the n-dimensional ADM mass

$$16\pi M_{ADM} = (n-1)(n-2)\Omega_{n-1}M \tag{1.33}$$

where Ω_{n-1} is the area of the unit $(n-1)$ sphere. For the five-dimensional case, the area of the unit sphere is $2\pi^2$ and hence it follows that our ADM mass is given by

$$M_{ADM} = \frac{3\pi}{4}M \tag{1.34}$$

Another key property of static black hole solutions is the surface gravity. If we were discussing the properties of non-relativistic objects this would be the acceleration required to keep stationary a test particle on the surface of the object [21]. However for the case of the relativistic black hole this surface is defined as the event horizon, and hence asymptotically at this event horizon the acceleration would appear to be infinite. For a general black hole, a formal definition of the surface gravity κ_s is given in terms of Killing fields [26]

$$\nabla_{\mu}(\xi^{\nu}\xi_{\nu}) = -2\kappa_s\xi_{\mu} \tag{1.35}$$

however we are only considering static solutions and hence for our purposes we can limit ourselves to a more intuitive solution. In this case the test particle experiences a force given by the tensor equation

$$\frac{f^{\mu}}{m} = \ddot{x}^{\mu} + \Gamma^{m}_{\nu\rho}\dot{X}^{\nu}\dot{X}^{\rho} = a^{\mu}. \tag{1.36}$$

For the extremal black hole, at the event horizon, the norm of this acceleration approaches infinity $a = \sqrt{g_{\mu\nu}a^{\mu}a^{\nu}} \xrightarrow[horizon]{} \infty$. Furthermore the redshift factor experienced by such a particle is defined as

$$V = \sqrt{|g_{tt}|} \tag{1.37}$$

and for an extremal black hole, $V \xrightarrow[horizon]{} 0$. The surface gravity κ_s is then simply the product of these two properties at the limit of the event horizon.

$$\kappa_s = \lim_{Horizon}(Va) \tag{1.38}$$

In the case of the four dimensional Reissner-Nordström black hole, it follows that the surface gravity is then given by

$$\kappa_s \pm = \pm\frac{r_+ - r_-}{2r^2_{\pm}}. \tag{1.39}$$

By restricting to a case of extremal black holes these two horizons coincide, and hence the surface gravity vanishes.

$$K_s|_{extremal} = 0 \tag{1.40}$$

This result can then be extended to higher dimensions such that we can observe that the five dimensional black hole also has zero surface gravity.

1.2.4 Laws of Black Hole Thermodynamics

One of the more interesting and surprising aspects of black holes, is that they appear to obey laws which have direct comparison with the classical laws of thermodynamics [21, 27]. The first of these laws is the zeroth law. In thermodynamics this law corresponds to a description of thermal equilibrium between different bodies, such that if three objects are in contact and $T_A = T_B$ and $T_B = T_C$ then $T_A = T_C$. This can equally be stated that for an object to be in thermal equilibrium it must have constant temperature. For the black holes which we are discussing, the surface gravity is directly comparable to this description of thermal equilibrium. This follows from κ_s remaining constant upon the horizon of the black hole. We can also look at the first law in thermodynamics which is relevant to the form of solutions we are interested in, and compare that to two infinitely close stationary black holes in Einstein gravity

$$\delta E = T\delta S + pdV + \mu N \tag{1.41}$$

$$\delta M = \frac{1}{8\pi}\kappa_s \delta A + \Omega\delta J + \mu\delta Q \tag{1.42}$$

where the terms involving the angular momentum J and rotation velocity Ω, are zero in our static case. If we had additional matter fields there would be additional terms to consider, however this direct comparison would seem to suggest the surface gravity and temperature can be identified. It also appears to suggest that the area of the event horizon, A, and the entropy, S, are connected and this leads to a definition of black hole entropy. For a classical black hole, one would expect the temperature to be zero, as no radiation should be able to escape. However quantum mechanics dictates that the black hole emits Hawking radiation, and hence a Hawking temperature can be defined

$$T_H = \frac{\kappa_S}{2\pi} \tag{1.43}$$

Also the Bekenstein-Hawking entropy of the black hole is proportional to the area as

$$S_{BH} = \frac{A}{4G_N}. \tag{1.44}$$

where we have set Boltzmanns constant to unity and reintroduced Newtons constant to show that the entropy by this definition is dimensionless as expected.

This definition of the entropy is reinforced by the second law of black hole thermodynamics, which states that [26]

$$\delta A \geq 0. \tag{1.45}$$

Hence, the area of the event horizon must be non-decreasing and hence justifying the relation between area and black hole entropy.

Finally there is the third law of thermodynamics, which can be stated as the entropy of extremal black holes vanishes. However, as there are thermodynamic counterexamples to $S \to 0$ as $T \to 0$ such as various forms of glass, a more instructive description is that it is impossible for $T \to$ to be reached by a physical process in finite time. This we provide without proof, as this would require an involved approach with quantum mechanics, either using the Euclidean path integral or through Minkowski canonical methods. Despite this, these laws of black hole physics are important concepts with consequence for our dimensionally lifted instanton solutions, as we will show they have an associated entropy we can calculate.

1.2.5 The attractor mechanism

Currently we have considered black holes with a line element given by

$$ds^2 = -e^{-2h(r)}dt^2 + e^{2k(r)}dr^2 + r^2 d\Omega^2 \tag{1.46}$$

where for simplicity we are considering the four dimensional case. However, for a static spherically symmetric solutions we can rewrite this into the form

$$ds^2 = -e^{2f(r)}dt^2 + e^{2g(r)}\left(dr^2 + r^2 d\Omega^2\right). \tag{1.47}$$

For extremal four dimensional Reissner-Nordström black holes we have the relation

$$f(r) = -g(r) \tag{1.48}$$

and hence has the form of the harmonic function

$$e^{2g(r)} = \left(1 + \frac{Q}{r^2} + \dots\right) \tag{1.49}$$

at leading order. This solution has two distinctive regimes. In the limit $r \to \infty$ it becomes asymptotically flat. That is $e^g \to 1$. However for $r \to 0$ we have the near horizon limit where

$$ds^2 = -\frac{r^2}{Q^2}dt^2 + \frac{Q^2}{r^2}dr^2 + Q^2 d\Omega^2 \tag{1.50}$$

We have now reduced the line element into the standard $AdS^2 \times S^2$ form with the area of the limiting sphere being the area of the event horizon $A = 4\pi Q^2$. It follows that this solution is interpolating between two vacua. The value of the scalar fields for large r can be arbitrarily chosen, however the values for $r \to 0$ are determined solely by the charges of the black hole solution. The term attractor refers to this behaviour, where we can show that by imposing a regular horizon solution then the black hole has a fixed point behaviour in the near horizon limit irrespective of the value of the scalar fields at infinity. More specifically, the attractor equations are required so that one can obtain a fully supersymmetric solution with the required $AdS^2 \times S^2$ geometry [28]

This is an important result that motivates the description of black hole solutions as solitons. That is two dimensional kink-solutions which interpolate between two vacua. However despite not requiring it for the models that we present, in order to fully understand the attractor behaviour we will now consider the role that supersymmetry has in describing an important subset of the solutions we will determine. In particular we will show that the Reissner-Nordström solutions can be described as supersymmetric solitons.

1.3 Supersymmetry

This thesis is focused upon the dimensional lifting of instanton solutions to black holes. The solutions that we would like to obtain include those that allow for multi-centred black hole solutions. In order for such solutions to exist and be stable then the forces between the constituents must vanish for arbitrary distance. Such examples would include the Majumdar-Parpapetrou solutions of Einstein Maxwell theory [29]. This cancellation of forces is often explain through imposing supersymmetry upon the model as this allows one to look for Killing spinors and hence stationary multi-centred solutions. Whilst our approach will allow us to generate solutions without such restrictions, we would therefore also like to investigate the supersymmetric subclass of models. Furthermore the type of models that we investigate in this context are lifted from the four dimensional $\mathcal{N} = 2$ vector multiplets [30], and hence we will also discuss the motivation for looking at such models. Finally we will show how the stationary black hole solutions that we have so far discussed and ultimately generate include those that can be labeled as supersymmetric solitons. [23]

1.3.1 Supersymmetry Algebra

Theories that are supersymmetric relate integer bosonic spin fields to half integer fermionic spin fields through a class of transformations called supersymmetry transformations generated by Q^A_α. According to [31] we can write the most general four-dimensional algebra associated with this transformation as

$$\{Q^A_\alpha, Q^{+B}_{\dot{\beta}}\} = 2\dot{\sigma}^\mu_{\alpha\dot{\beta}} P_\mu \delta^{AB} \quad (1.51)$$

$$\{Q^A_\alpha, Q^B_\beta\} = 2\epsilon_{\alpha\beta} Z^{AB} \quad (1.52)$$

where we have the supersymmetries counted by the index $A = 1 \ldots \mathcal{N}$ and the index $\alpha = 1, 2$ is a spinor index. Hence we also have the hermitian conjugate $Q^{+B}_{\dot{\beta}}$. Alternatively this could have been stated in terms of 4 component Majorana spinors, however our discussion remains unchanged [32].

The matrix Z^{AB} is antisymmetric and is composed of operators that commute with all the operators in the super Poincaré algebra. These operators are therefore referred to as central charges, and are present in extended supersymmetric theories ($\mathcal{N} > 1$). The case of $\mathcal{N} = 2$ is of particular interest to us. For massive representations, the momentum vector P_μ can be brought to the form $P_\mu = (-M, \vec{0})$. This can be substituted into the algebra and by setting $2|Z| = |Z^{12}|$ we obtain the algebra [33]

$$\begin{aligned} \{Q^A_\alpha, Q^{+B}_{\dot{\beta}}\} &= 2M\delta_{\alpha\dot{\beta}}\delta^{AB} && (1.53) \\ \{Q^A_\alpha, Q^B_\beta\} &= 2|Z|\epsilon_{\alpha\beta}\epsilon^{AB} && (1.54) \end{aligned}$$

It is then possible to rewrite the algebra as fermionic creation and annihilation operators by taking linear combinations of the supersymmetry charges

$$\begin{aligned} \{a_\alpha, a^+_\beta\} &= 2(M + |Z|)\delta_{\alpha\beta} && (1.55) \\ \{b_\alpha, b^+_\beta\} &= 2(M - |Z|)\delta_{\alpha\beta}. && (1.56) \end{aligned}$$

This allows for the construction of the representation of the algebra. However, quantum mechanics requires us to only consider unitary representations, and hence implies an important bound upon the mass

$$M \geq |Z| \qquad (1.57)$$

which is called the BPS bound. It then follows that there are two separate classes to consider

For $M > |Z|$ we obtain the unitary representation of the algebra with 4 creation operators hence and dimension 2^4. Alternatively, the bound can be saturated and therefore $M = |Z|$. Subsequently the representation contains null states which have to be divided out to ensure a unitary representation. This is realised by setting all b-operators such that they act trivially on the representation and hence remain invariant under half of the supersymmetry transformations. It therefore follows that the remaining creation operators generate a representation of dimension $2^2 = 4$ of the supersymmetry algebra. It is then worth noting that massless representations can then be formally obtained by considering $M = Z = 0$ however massless representations should be considered separately as their is no rest frame and hence the standard form of the momentum vector P_μ no longer applies. However, as in the BPS case one sets half the creation and annihilation operators to zero to obtain representations with $2^2 = 4$ states.

The field content of the appropriate physical theory can be identified by investigating the elementary particles which correspond to the representations of the Poincaré group, and hence we also need to consider the supersymmetric extension of the Poincaré Lie algebra. The massive representations are classified according to the mass and spin, $s = 0, \frac{1}{2}, 1, \ldots$. The 'little group' which is classified by this spin, is then the $SO(3)$ subgroup which leaves the momentum $P_\mu = (-M, \vec{0})$ invariant. However for massless representations, these are classified according to helicity which classifies the 'little group' $SO(2)$ of massless particles. This then leaves the momentum $P_\mu = (E, 0, 0, \pm E)$ invariant. Since this is an abelian group it has a one-dimensional irreducible representation and the possible helicity values are $\lambda = 0, \pm\frac{1}{2}, \pm 1, \ldots$

In order to obtain a representation of the full Poincaré superalgebra then we have to assign either spin (for massive representations) or helicity (for massless representations) to the ground state. Thus the particle content of BPS representations is then easily obtained by observing that for both the BPS and massless cases half of the creation and annihilation operators have to be set to zero. Thus BPS states can be understood as giving mass to massless representations without altering the total number of states. For this thesis we are interested in the BPS vector multiplets and hypermultiplets.

In the case of massless vector multiplets we can choose the ground state to have helicity $\lambda = 0$. Hence by applying the two creation operators available we also obtain two states with helicity $\lambda = \frac{1}{2}$ and one with $\lambda = +1$. These four states then form the representation of the supersymmetry algebra. However these two states cannot be physical on their own as the $\lambda = +1$ state must belong to a massless vector boson, which also has a negative polarisation state of $\lambda = -1$ due to invariance under CP-transformations. Therefore the massless vector multiplet is obtained by adding a second representation with a ground state of $\lambda = -1$, two states of $\lambda = -\frac{1}{2}$ and one state of $\lambda = 0$. The physical massless vector multiplet therefore contains one massless vector with $\lambda = \pm 1$, two Weyl spinors and one complex scalar. It therefore has four fermionic and four bosonic degrees of freedom. The BPS vector multiplet is a massive multiplet with the same number of degrees of freedom, and hence is called the short vector multiplet. The transition between the massless and massive representations occurs through a Higgs mechanism where one of the two scalars becomes the longitudinal mode of a massive vector boson with the three polarisation states $s_z = 1, 0, -1$. The spin content of the BPS vector multiplet is then $(1[1], 2[\frac{1}{2}], 1[0])$. We discuss the five dimensional version of the massless vector multiplet within the next section, as this plays a key role in connecting our discussions to BPS black holes that have been discussed in the literature.[34]

In the hypermultiplet case we can also start by considering massless states. However in this case we start with a state of helicity $\lambda = -\frac{1}{2}$ and hence obtain two states of $\lambda = 0$ and one of $\lambda = \frac{1}{2}$ This is sometime called the half-multiplet and it could describe a single Weyl spinor with two real scalars. However, as the physical theory requires the anti-particle states this would require the half-hypermultiplet to be self-conjugate and hence not carry any type of charge. To account for charge, a second half-multiplet is added to the first and thus we obtain a multiplet with two Weyl spinors and four real scalars. BPS hypermultiplets are therefore massive versions of this hypermultiplet. This is important to this thesis, not in terms of the Lagrangian fields but in our construction of black hole solutions. These solutions are invariant under half of the supersymmetry transformations and under quantisation correspond to solitonic excitations of the hypermultiplet theory [23]. These black hole hypermultiplets therefore provide an extension to the underlying theory, which is described by the gravity and vector multiplets. It is therefore important for us to understand how the models we investigate in this thesis connects to this underlying vector multiplets, and hence allow us to construct these BPS black hole solutions.

1.3.2 Vector Multiplets

Throughout this thesis we are interested in instanton solutions and black holes that come from considering the properties of non-linear sigma models. Generally our solutions will not require supersymmetry, however we would also like to see how these models relate to the supersymmetric case. To achieve this we consider the five dimensional vector multiplets of $\mathcal{N} = 2$ supersymmetry as for which an off shell realization was given in [35, 30].

The building blocks for the vector multiplet in 5 Dimensional N=2 SUSY, are the vector field A_μ, a pair of symplectic Majorana spinor fields λ^i, the scalar σ and the auxiliary tensor field Y^{ij} of the R-Symmetry group $SU(2)$. We take N copies of this vector multiplet, then the kinetic terms have form

$$\mathcal{L} = \left(-\frac{1}{4}F^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}\bar{\lambda}^I \not{\partial}\lambda^J - \frac{1}{2}\partial_\mu\sigma^I\partial^\mu\sigma^J + Y^{Iij}Y^J_{ij}\right) a_{IJ}(\sigma) \qquad (1.58)$$

where $I, J \in \{1, ..., N\}$ and this is now a non-linear sigma model for the scalars, as the matrix $a_{IJ}(\sigma)$ depends upon the scalar field σ.

This model is then invariant under the transformations

$$\begin{aligned} \delta\sigma^I &= \frac{i}{2}\bar{\epsilon}\lambda^I \\ \delta A^I_\mu &= \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^I \\ \delta\lambda^{Ii} &= -\frac{1}{4}\gamma^{\mu\nu}F^I_{\mu\nu}\epsilon^i - \frac{i}{2}\not{\partial}\sigma^I\epsilon^i - Y^{Iij}\epsilon_j \\ \delta Y^{Iij} &= -\frac{1}{2}\bar{\epsilon}^{(i}\not{\partial}\lambda^{j)I} \end{aligned} \qquad (1.59)$$

This invariance can be observed by first separating the Lagrangian in its four appropriate parts and then finding the variation of each part individually

$$\mathcal{L} = -\mathcal{L}_A - \mathcal{L}_\sigma - \mathcal{L}_Y - \mathcal{L}_\lambda \tag{1.60}$$

The Majorana condition for the fermionic field is also required. This is given by

$$\begin{aligned} \epsilon^i &= (i\gamma_5 C)\epsilon^{ij}\bar{\epsilon}_j \quad C = i\gamma_0 \\ &= -\gamma_5\gamma_0\bar{\epsilon}^i \end{aligned} \tag{1.61}$$

Firstly, using the antisymmetry of $F^{\mu\nu}$

$$\begin{aligned} \delta\mathcal{L}_A &= \frac{1}{2}F^{\mu\nu}\delta F_{\mu\nu} \\ &= \frac{1}{2}F^{\mu\nu}(\partial_\mu\bar{\epsilon}^i\gamma_\nu\lambda_i - \partial_\nu\bar{\epsilon}^i\gamma_\mu\lambda_i) \\ &= -F^{\mu\nu}\bar{\epsilon}^i\gamma_{[\mu}\partial_{\nu]}\lambda_i \end{aligned} \tag{1.62}$$

this should now cancel with the term from $\delta\mathcal{L}_\lambda$ involving λ_i. This first requires $\delta\bar{\lambda}^i$ to be calculated. Concentrating on the relevant term

$$\begin{aligned} \delta\bar{\lambda}^i &= \frac{1}{2}F^{\mu\nu}\bar{\gamma}_{\mu\nu}\epsilon_i \\ &= -\frac{1}{2}F^{\mu\nu}(\epsilon_i)^\dagger(\gamma_{\mu\nu})^\dagger\gamma_5\gamma_0 \end{aligned} \tag{1.63}$$

Using $(\gamma_0)\dagger = -\gamma_0$ and $(\gamma_i)\dagger = \gamma_i$, it can be seen that $\gamma_{cd}\gamma_5\gamma_0 = \gamma_5\gamma_0\gamma_{dc}$

$$\begin{aligned} \delta\bar{\lambda}^i &= -\frac{1}{2}F^{\mu\nu}(\epsilon_i)\dagger\gamma_5\gamma_0\gamma_{\nu\mu} \\ &= \frac{1}{2}F^{\mu\nu}\bar{\epsilon}\gamma_{\nu\mu} \\ &= -\frac{1}{2}F^{\mu\nu}\bar{\epsilon}\gamma_{\mu\nu} \end{aligned} \tag{1.64}$$

Hence

$$\begin{aligned} \delta\mathcal{L}_\lambda &= \frac{1}{2}(\delta\bar{\lambda}^i\partial\!\!\!/\lambda_i + \bar{\lambda}^i\partial\!\!\!/\delta\lambda_i) \\ &= \frac{1}{2}\left(F^{\mu\nu}\bar{\epsilon}^i\gamma_{\mu\nu}\gamma^\rho\partial_\rho\lambda_i - \bar{\lambda}^i\gamma^\rho\partial_\rho F^{\mu\nu}\gamma_{\mu\nu}\epsilon_i\right) \\ &= \frac{1}{2}\left(-F^{\mu\nu}\bar{\epsilon}^i\gamma_{\mu\nu}\gamma^\rho\partial_\rho\gamma_i + \partial_\rho\bar{\lambda}^i\gamma^\rho F^{\mu\nu}\gamma_{\nu\mu}\epsilon_i\right) \end{aligned} \tag{1.65}$$

Using $\bar{\lambda}^i = -\gamma^0\gamma^5\lambda_i$ the second term can be put into the form $-F^{\mu\nu}\bar{\epsilon}^i\gamma_{\mu\nu}\gamma^\rho\partial_\rho\lambda_i$ and the two terms can be combined to give:

$$\delta\mathcal{L}_\lambda = -F^{\mu\nu}\bar{\epsilon}^i\gamma_{\mu\nu}\gamma^\rho\partial_\rho\lambda_i \tag{1.66}$$

We can decompose $\gamma_{\mu\nu}\gamma^\rho$ as

$$\begin{aligned} \gamma_{\mu\nu}\gamma^\rho &= \frac{1}{2}\left[\gamma_{\mu\nu}, \gamma^\rho\right] - \frac{1}{2}\left\{\gamma_{\mu\nu}, \gamma^\rho\right\} \\ &= -\frac{1}{2}\left(2\delta^\rho_\mu\gamma_\nu - 2\delta^\rho_\nu\gamma^\mu\right) \\ &= -\delta^\rho_{[\mu}\gamma_{\nu]} \end{aligned} \tag{1.67}$$

where the second term in the first line vanishes using the Bianchi Identity

$$\epsilon_{\mu\nu\rho\sigma}\partial^{\nu}F^{\rho\sigma} = 0 \tag{1.68}$$

This implies that

$$\begin{aligned} \delta\mathcal{L}_{\lambda} &= F^{\mu\nu}\bar{\epsilon}^{i}\delta^{\rho}_{[\mu}\gamma_{\nu]}\partial_{\rho}\lambda_{i} \\ &= F^{\mu\nu}\bar{\epsilon}^{i}\gamma_{[\mu}\partial_{\nu]}\lambda_{i} \end{aligned} \tag{1.69}$$

Hence $-\delta\mathcal{L}_{A} - \delta\mathcal{L}_{\lambda} = 0$ for the spinor terms, as required. The invariance of the scalar and tensor terms, are similarly observed.

This supersymmetric invariance applies to the model with arbitrary numbers of vector multiplets, to within a derivative or variation upon the metric. When the metric is independent of σ this implies that the theory maintains its invariance. However this is not the most general form of the theory, as additional terms can be added to the action whose supersymmetry variations cancel such derivative terms. Through adding interaction terms of a Chern-Simons type, and requiring $\frac{\partial}{\partial\sigma^{K}}a_{IJ}(\sigma)$ to be symmetric in all 3 indices the metric can then be written as the second derivative of a prepotential $F(\sigma)$.

Using this prepotential, the derivative terms of the metric can be rewritten as

$$F_{IJK}(\sigma) := \frac{\partial}{\partial\sigma^{I}}\frac{\partial}{\partial\sigma^{J}}\frac{\partial}{\partial\sigma^{K}}F(\sigma) \tag{1.70}$$

This allows us to add interaction terms to the Lagrangian and we obtain the general 5 Dimensional Lagrangian:

$$\begin{aligned} \mathcal{L} = &\left(-\frac{1}{4}F^{I}_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}\bar{\lambda}^{I}\not{\partial}\lambda^{J} - \frac{1}{2}\partial_{\mu}\sigma^{I}\partial^{\mu}\sigma^{J} + Y^{Iij}Y^{J}_{ij}\right)a_{IJ}(\sigma) \\ + &\left(-\frac{1}{24}\epsilon^{\mu\nu\lambda\rho\sigma}A^{I}_{\mu}F^{J}_{\nu\lambda}F^{K}_{\rho\sigma} - \frac{i}{8}\bar{\lambda}^{I}\gamma^{\mu\nu}F^{J}_{\mu\nu}\lambda^{K} - \frac{i}{2}\bar{\lambda}^{iI}\lambda^{jJ}Y^{K}_{ij}\right)F_{IJK} \end{aligned} \tag{1.71}$$

We have currently only considered supersymmetry invariance, however it is also important that any physical model we discuss is also gauge invariant. This provides a constraint on the prepotential such that it must be at most a polynomial of degree 3. [30] This ensures that the Chern-Simons terms are gauge invariant up to partial integration. Any higher order polynomials would generate non trivial terms that break the gauge invariance of the action. Since the remaining supersymmetry variations are proportional to the fourth derivative of the prepotential, the action is also supersymmetric once this condition is imposed. Hence this provides important consequences upon the form of the prepotential we can choose if we restrict ourselves to purely supersymmetric models.

1.3.3 BPS Black Holes

Given a supersymmetric theory there will be field configurations of which some will be invariant under part of the supersymmetry variations and saturate the BPS bound. In the case of vector multiplets we have seen that it is a $\frac{1}{2}$-BPS multiplet where half the supertransformations act trivially. If these field configurations then also solve the field equations then by our definitions they are then supersymmetric solitons. By now coupling the theory to gravity, we obtain supergravity such that the supersymmetry parameter now depends on space-time. Hence we can immediately observe that this BPS condition can be applied to the solitons of gravitational theories which we have already discussed include black holes solutions. The BPS conditions requires the existence of a spinor field which generates a supertransformation under with the soliton is invariant. These spinor fields are the Killing spinors, and analogous to the concept of Killing vectors. [28]

The four dimensional extremal Reissner-Nordström black hole is a $\frac{1}{2}$ BPS solution of $\mathcal{N} = 2$ supergravity. Whilst the inclusion of additional vector multiplets allow more general $\frac{1}{2}$ BPS solutions with non-constant scalar fields, we can now concentrate upon the four dimensional supersymmetric extremal Reissner-Nordström black hole and connect it to the attractor mechanism.

We have already highlighted that the black hole metric can be brought to an $AdS^2 \times S^2$ form (1.50) by enforcing a static and spherically symmetric solution. By imposing that this also extends to the gauge fields and scalar field of the of the supersymmetric model, then these can be written as depending only upon the radial coordinate r. It then also follows that the field strength tensor F_{mn} has only two independent components.

The charges carried by the solutions are then defined by the flux integrals over the asymptotic 2-sphere

$$(p^I, q_I) = \frac{1}{4\pi}\left(\oint F^I, \oint G_I\right) \tag{1.72}$$

where F, G are the two-forms associated with (F_{mn}) where $G_I = *F^I$ and p^I, q_I are the magnetic and electric charges respectively. It follows that these transform as vectors under symplectic transformations and hence contracting with the scalars one obtains the relation [28]

$$Z = p^I F_I - q_I X^I. \tag{1.73}$$

This Z is often called the central charge despite being a function of r. However, this Z field is related to the central charge of the supersymmetry that we have discussed by taking the asymptotically flat limit that $r \to \infty$. When this limit is taken, this computes the electric and magnetic charge of the graviphoton and combines it into the complex central charge. The central charges of the supersymmetry then relates to the mass of the black hole

$$M = |Z|_\infty \tag{1.74}$$

which depends on the charges and the values of the scalar fields within the model at infinity. Hence BPS black holes saturate the mass bound implied by the supersymmetry algebra.

The event horizon provides a second asymptotic region where the solution must be fully supersymmetric for a regular horizon. In the near horizon limit, the line element is now

$$ds^2 = -\frac{r^2}{|Z|^2_{hor}}dt^2 + \frac{|Z|^2_{hor}}{r^2}dr^2 + |Z|^2_{hor}d\Omega \tag{1.75}$$

with $|Z|^2_{hor}$ the value of $|Z|^2$ at the horizon. It follows that area of the event horizon is $A = 4\pi|Z|^2$ and it also can be shown that the entropy depends upon the central charge

$$S = \pi|Z|^2_{hor} \tag{1.76}$$

where the value of S is now determined by only the charges. It is now possible to write down attractor equations [36, 37] which express the horizon values of the scalar fields in terms of the charges

$$\begin{pmatrix} \bar{Z}X - Z\bar{X} \\ \bar{Z}F(X) - Z\bar{F}(X) \end{pmatrix} = i\begin{pmatrix} p \\ q \end{pmatrix} \tag{1.77}$$

This behaviour is due to requiring a fully supersymmetric solution with $AdS^2 \times S^2$ geometry with non vanishing gauge fields [38] .

By requiring the solutions to obey the attractor equations, we observe that the BPS black holes are determined through their charges, and hence through the central charges within the supersymmetric theory. These properties have been investigated in detail in the literature [39, 40], however our intention is to show that we can find similar properties by considering instantons from a general four dimensional model and lifting those solutions to five dimensional black holes.

This chapter has introduced the concept of an instanton and soliton solution within the context of black holes and the models for which we will investigate within the forthcoming chapters. We have also introduced the connection between the BPS black hole solutions and supersymmetric $\mathcal{N} = 2$ vector multiplets. In the forthcoming chapters we aim to generate four dimensional instanton solutions which can be lifted to five dimensional black holes without requiring the constraint of supersymmetry. Through constructing these solutions we develop a in depth analysis of the various approaches that avoid the problems of Derrick's Theorem and show that our instanton solutions are valid structures that provide contributions to transition amplitudes. By lifting the solutions to five dimensions, we then investigate the properties of the black hole solutions that we generate.

Chapter 2

Instantons and Harmonic Maps

2.1 Sigma Models and Dimensional Reduction

The main interest for this thesis is the construction of four dimensional instanton solutions, and their subsequent lifting to black holes. Therefore the bulk of what is presented here is based upon the published paper, "Instantons, Black Holes and Harmonic Functions" [41] where we look at sigma models of the form

$$S[\sigma, b]_{(0,4)} = \int d^4x N_{IJ} \left(\partial_m \sigma^I \partial^m \sigma^J - \partial_m b^I \partial^m b^J\right). \tag{2.1}$$

This is a four dimensional, Euclidean model in flat space-time, E, with indices $m = 1, 2, 3, 4$. It contains non trivial scalar fields, σ^I and the axionic b^I and hence our target space, M, is $2n$-dimensional with $I = 1, \ldots, n$. N_{IJ} is real, positive definite and depends only on σ^I and hence the metric on M has n-commuting isometries which shift the axionic scalars b^I

$$b^I \to b^I + C^I \tag{2.2}$$

where C^I are constants. It is also worth noting at this stage that the minus sign between the kinetic terms in our action implies the metric $N_{IJ} \oplus (-N_{IJ})$ of M has a split signature (n, n) which is consistent with the approach of defining Euclidean actions from Wick rotations that includes an analytic continuation of the axionic field $b^I \to ib^i$. This includes the instanton solutions from string theory and supergravity including the D-Instanton of type IIB string theory presented in the paper [5] by Gibbons, Green and Perry

This D-Instanton is a solution to the ten dimensional lagrangian

$$\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial a)^2 \tag{2.3}$$

with two scalar fields, the dilaton ϕ and an $R \times R$ scalar a. By Wick rotating this lagrangian and imposing a constraint of $a \to \alpha = ia$ they construct an instanton solution which preserves half the total supersymmetries. This constraint is imposed without justification, and provides the focus of our aims within this thesis. By taking an identical approach for our four dimensional lagrangian, we provide an analytic justification for this constraint and also investigate other approaches that would allow us to construct an instanton solution.

For simplicity we have restricted the lagrangian of our model to four dimensions. Furthermore the four dimensional lagrangian that we are investigating is the dimensionally reduced form of a $(1+4)$ dimensional non-linear sigma model. Consequently we expect the four dimensional instanton solutions to correspond to soliton solutions in five dimensions. This will allow us to connect our solutions to the properties that are already understood for five dimensional soliton solutions such as black holes. In order to understand this connection we will begin the discussion with the $(1+4)$ dimensional lagrangian.

2.1.1 Dimensional Reduction and Wick Rotations

Initially we are interested in the 1+4 dimensional theory without coupling to gravity

$$S[\sigma, A, \dots]_{(1,4)} = \int d^5x \left(-\frac{1}{2} N_{IJ}(\sigma) \partial_\mu \sigma^I \partial^\mu \sigma^J - \frac{1}{4} N_{IJ}(\sigma) F^I_{\mu\nu} F^{J|\mu\nu} + \dots \right). \tag{2.4}$$

where we have five dimensional greek indices $\mu, \nu, \dots = 0, 1, 2, 3, 4$, a n-dimensional target space M_r with positive definite metric N_{IJ} and the abelian field strength $F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu$. We can then reduce this to our four dimensional action by restricting to static and purely electric five dimensional backgrounds. By setting

$$\partial_0 \sigma^I = 0, \quad \partial_0 A^I_m = 0, \quad F^I_{mn} = 0, \tag{2.5}$$

identifying $b^I = A^I_0$, and then dropping the integration over time it can easily be seen that (2.4) reduces to (2.1) with a conventional minus sign difference in front of the overall action. The additional terms that are included within the dots in (2.4) then include any terms that do not contribute to static five dimensional configurations involving scalars and gauge fields. As we are also only going to consider electrical gauge configurations this allows us to ignore, for example, Chern-Simons and fermionic terms within our analysis.[19]

An alternative approach to recovering a four-dimensional Euclidean action from our five dimensional model is to first reduce the action in a space-like direction as opposed to a time-like one, and then perform a Wick rotation to obtain the euclidean action. A reduction of (2.4) in a space-like direction obtains

$$S[\sigma, b]_{(1,3)} = -\int d^4x \frac{1}{2} N_{IJ}(\sigma) \left(\partial_{\bar{m}} \sigma^I \partial^{\bar{m}} \sigma^J + \partial_{\bar{m}} b^I \partial^{\bar{m}} b^J \right). \tag{2.6}$$

where $\bar{m} = 0, 1, 2, 3$. This action has a $2n$-dimensional target space M' with positive definite metric $N_{IJ} \oplus N_{IJ}$. We can then Wick rotate to obtain the positive definite

Euclidean action

$$S[\sigma, b]_{(0,4)} = -\int d^4x \frac{1}{2} N_{IJ}(\sigma) \left(\partial_m \sigma^I \partial^m \sigma^J + \partial_m b^I \partial^m b^J\right). \tag{2.7}$$

which also has a $2n$-dimensional target space M' with positive definite metric $N_{IJ} \oplus N_{IJ}$. By comparing (2.7) to (2.1) we see that these two actions differ through a minus sign and hence the split signature metric of (2.1). However, by following the methods in the literature such as [5] we can relate them through performing the analytic continuation $b^I \to ib^I$. Our two actions that we have obtained through time-like and space-like reductions are therefore related through a modified Wick rotation which act non-trivially on the axionic scalars.

2.2 Equations of Motion and Harmonic Maps

In order to investigate the properties of these sigma models we need to develop an approach to solve their equations of motion. We perform this by following the method in section of 9 "BPS Branes in Supergravity" by Stelle [18]. In this case solving our equations of motion is equivalent to constructing a harmonic map from the space-time manifold X to the (pseudo)Riemannian scalar target space M. In our case, we are interested in models where X is flat Euclidean space, and hence we have a general action

$$S[\Phi]_{(0,4)} = \int d^4x N_{ij}(\Phi) \partial_m \Phi^i \partial^m \Phi^j \tag{2.8}$$

and hence we have equations of motion

$$\triangle \Phi^i + \Gamma^i_{jk} \partial_m \Phi^j \partial^m \Phi^k = 0 \tag{2.9}$$

with the Christoffel symbols Γ^i_{jk}, of the metric N_{ij} of M. Hence we have the coordinate form of the equation of a harmonic map $\Phi : E \to M$ from the Euclidean space E to the target space M. We would like to find harmonic solutions to the equations of motion, and hence are interested in finding a criteria that ensures these exist.

Defining a submanifold $N \subset M$ that is totally geodesic, i.e. every geodesic of N is also a geodesic of M, we reduce our problem to finding the harmonic map $\phi : E \to N \subset M$ as any harmonic map $E \to N$ along with a geodesic map $N \to M$ is also harmonic [17]. We are interested in reducing (2.9) to a form which guarantees that we can find harmonic solutions. This occurs when our submanifold N is flat, and hence the Christoffel symbols vanish if we use affine coordinates. Consequently we parametrise our scalar fields such that the independent scalars ϕ^a, with $a = 1, \ldots, \dim N$ are the affine coordinates on N and the equation of motion reduces to

$$\triangle \phi^a = 0. \tag{2.10}$$

If N has dim $N <$ dim M, then the remaining harmonic fields can be expressed in terms of the solution for ϕ^a.

We can now apply this to our four dimensional Euclidean action, which is para-Hermitian and has n commuting shift symmetries. Through varying the action (2.1) with respect to the two real fields that we have σ^I, and b^I we obtain

$$\partial^m \left(N_{IJ}\partial_m\sigma^J\right) - \frac{1}{2}\partial_I N_{JK}\left(\partial_m\sigma^J\partial^m\sigma^K - \partial_m b^J\partial^m b^K\right) = 0 \tag{2.11}$$

$$\partial^m \left(N_{IJ}\partial_m b^J\right) = 0 \tag{2.12}$$

This then has a drastic simplification if we require the relation

$$\boxed{\partial_m\sigma^I = \pm\partial_m b^I.} \tag{2.13}$$

This is an important constraint that we impose upon our model throughout this thesis and relates the scalar field with the axionic one. We will refer to this as the extremal instanton ansatz and it corresponds geometrically to restricting the scalar fields to only vary along the null directions of the metric of M such that they take values in the submanifold $N \subset M$ which is completely isotropic. It is this constraint than now allows us to find our instanton solutions, including multi-centred solutions without requiring supersymmetry.

Consequently our equations of motion collapse into the single equation

$$\partial^m \left(N_{IJ}\partial_m\sigma^J\right) = 0 \tag{2.14}$$

It is worth noting at this point that our extremal instanton ansatz (2.13) is a sufficient but not necessary condition for reducing our equations of motion. Providing the metric N_{IJ} is invariant under the transformations

$$N_{IJ} \to N_{KL}R^K_I R^L_J \tag{2.15}$$

with R^I_J a constant matrix, then we can see that our equations of motion still reduce to (2.14). This corresponds to an isometry of $N_{IJ} \oplus (-N_{IJ})$ where

$$\sigma^I \to \sigma^I \quad , \quad b^I \to R^I_J b^J. \tag{2.16}$$

and has occurred in other literature in the context of extremal black hole solutions where $R^I_J \neq \delta^I_J$ corresponds to non-BPS black holes [42, 43] and hence provides a distinction in how BPS and non-BPS extremal solutions can be understood geometrically. For this analysis we will impose our extremal instanton ansatz and hence restrict ourselves to the case of BPS solutions.

After imposing the extremal instanton ansatz (2.13) we obtained a reduced form of the equation of motion (**??**) Provided there exists dual fields σ_I with the property

$$\partial_m \sigma_I = N_{IJ} \partial_m \sigma^J \tag{2.17}$$

we can then reduce this further to a set of n harmonic equations. We do this by imposing the integrability condition

$$\partial_{[n}(N_{IJ}\partial_{m]}\sigma^J) = 0 \tag{2.18}$$

upon our equation of motion. Using the relation $\partial_{[n}\partial_{m]}\sigma^J = 0$ this is then equivalent to

$$\partial_{[n} N_{IJ} \partial_{m]} \sigma^J = \partial_K N_{IJ} \partial_{[n} \sigma^K \partial_{m]} \sigma^J = 0 \tag{2.19}$$

There are two possible approaches to solving this equation, and we will start by considering the constraints that we can apply to the form of the solution we expect to obtain.

2.2.1 Single Centred Solutions

The first approach to solving the integrability condition is to restrict the form of the solutions $\sigma^I(x)$ whilst making no assumptions about the form of the metric N_{IJ}. By assuming that the solution only depends upon one coordinate of our Euclidean space, E then (2.19) is solved. The simplest and most natural assumption we can make is to assume our solution is spherically symmetric $\sigma^I = \sigma^I(r)$, with radial coordinate r. This is a sensible choice of solution as we would like solutions that asymptotically approach ground states σ^I_{vac} =constant at infinity. We can therefore write the solution to the equation of motion $\triangle\sigma_I = 0$ as a single centred harmonic function

$$\sigma_I = H_I(r) = h_I + \frac{q_I}{r^2}. \tag{2.20}$$

This solution geometrically corresponds to a situation where the scalar fields flow along null geodesic curves in M, and we interpret this as a single centered instanton. Whilst it is clearly seen that h_I specifies the values of our dual field σ_I at infinity, we still require an interpretation for the q_I's. These will be shown to be understood as charges of an instanton located at $r = 0$ and this interpretation will be discussed later.

2.2.2 Multi-centred Solutions

Alternatively our second strategy involves making no assumptions about the solution, but instead looks at the conditions we can impose upon the metric. We would like to produce more general multi-centred solutions and hence cannot rely upon spherical symmetry as a simplification. Instead we can still solve the integrability condition by imposing the condition

$$\partial_{[K}N_{I]J} = 0 \tag{2.21}$$

upon the scalar metric.

To understand why this allows us to construct such solutions we can see that (2.21) is equivalent to requiring the first derivatives of $\partial_K N_{IJ}$ to be symmetric, or similarly the Christoffel symbols of the first kind $\Gamma_{IJ|K}$ to be completely symmetric. We know from the Poincaré Lemma that for a vector field V_I, if $\partial_{[k}V_{I]} = 0$, we can then write the vector field as the divergence of a scalar potential $V_I = \partial_I F$. Since our metric N_{IJ} is symmetric, we can therefore apply the Poincaré Lemma twice and see that it follows that (2.21) is equivalent to defining the metric as the second derivative of a Hesse Potential $\mathcal{V}$

$$N_{IJ} = \frac{\partial^2 \mathcal{V}}{\partial\sigma^I \partial\sigma^J} \tag{2.22}$$

However this presents the question of whether this has a geometrical or coordinate invariant meaning. The metric we have defined is a second rank tensor, however the second derivatives of a function are not. To obtain such a tensor field we have to apply a covariant derivative in the second step, $N_{IJ} = \nabla_I \partial_J \mathcal{V}$, however this connection can be different to the Levi-Civita connection. Denoting the connection symbols as γ^K_{IJ} to distinguish them from the standard Christoffel symbols Γ^K_{IJ}, we can show that this has a coordinate invariant meaning provided a connection ∇_I with certain properties exists.

We recall that the connection is torsion free if the connection symbols are symmetric $\gamma^K_{IJ} = \gamma^K_{JI}$. This is equivalent to imposing that

$$\nabla_I \nabla_J f = \nabla_J \nabla_I f \tag{2.23}$$

for all functions f [26]. Furthermore a connection is flat if the corresponding curvature tensor vanishes. For a torsion-free connection this is equivalent to imposing

$$\nabla_I \nabla_J v^k = \nabla_J \nabla_I v^K \tag{2.24}$$

for all vector fields v^k. For such flat and torsion free connections then we can find so-called affine coordinates where $\gamma^K_{IJ} = 0$ and hence $\nabla_I = \partial_I$ in any local coordinate patch. The connection symbols transform as a tensor under the affine transformation

$$\sigma^I \to A^I_J \sigma^J + B^J. \tag{2.25}$$

Imposing that a torsion free and flat connection exists on a manifold it therefore follows that the manifold can be covered by affine coordinate patches such that $\nabla_I = \partial_I$ on each patch. Consequently it follows that our definition of the metric (2.22) makes sense provided the fields σ^I correspond to ∇-affine coordinates.

However by restricting ourselves to the Levi-Civita connection we would be limited to the trivial case of flat N_{IJ} metrics. Whilst this suggests we would need to formulate our integrability condition in a coordinate independent way, by requiring σ^I to be ∇-affine coordinates then the coordinate invariant version of (2.21) is

$$\nabla_{[K}N_{I]J} = 0 \tag{2.26}$$

or equivalently that the third rank tensor $\nabla_K N_{IJ}$ is completely symmetric. We therefore obtain the mathematical definition of a Hessian manifold from [44], and have shown these properties of the connection to be equivalent to our integrability condition which was used to obtain (2.22).

Following from this definition we then observe that our integrability condition (2.19) is solved by defining a dual coordinate

$$\partial^m \partial_m \sigma_I = 0 \tag{2.27}$$

This is always possible given a metric defined from the Hessian potential $\mathcal{V}(\sigma)$ such that

$$\sigma_I \simeq \frac{\partial \mathcal{V}(\sigma)}{\partial \sigma^I} \tag{2.28}$$

Thus by imposing our metric to be Hessian we can find multi-centred instanton solutions of the form

$$\boxed{\sigma_I(x) = H_I(x) = h_I + \sum_{a=1}^{N} \frac{q_{aI}}{|x - x_a|^2}} \tag{2.29}$$

where h_I, q_{aI} are constants and x,$x_a \in E$ This corresponds to N instantons, with charges q_{aI} located at positions x_I.

2.3 Behaviour of Instanton Solutions

We have shown that instanton solutions can be found provided we restrict our metric N_{IJ} to being the second derivative of a Hesse Potential $\mathcal{V}$. However not all possible Hesse potentials will then provide solutions with a finite action when evaluated over the instanton solution. By investigating the behaviour of the solutions at their centres we can see if the solutions exhibit attractor behaviour such that the asymptotics are independent of any boundary conditions imposed at infinity and hence determined only by the charges. This fixed point behaviour ensures a finite action, and hence provides a collection of models which we can then investigate further. In particular two types of Hesse potential allow for a complete analysis, homogenous functions and the logarithms of homogenous functions, with the second corresponding to the models which can be lifted to five dimensions with gravity. We will also discuss the lifting of other models to five dimensions, with varying success and interest.

2.3.1 Hesse Potential $\mathcal{V} = \sigma^p$

The simplest model we can start with has a Hesse potential that depends on a single scalar σ and is homogenous of degree $p = N + 2$. Hence our metric, being the second derivative of the Hesse potential, is proportional to σ^N and our sigma model has the form

$$S = \frac{1}{2}\int d^4x\sigma^N\left(\partial_m\sigma\partial^m\sigma - \partial_m b\partial^m b\right). \tag{2.30}$$

There are a number of different interesting cases to consider here. The first case, $N = 0, p = 2$ corresponds to a free theory and hence is a trivial model to consider. However the case $N = 1, p = 3$ corresponds to Euclidean vector multiplets that are obtained by the time reduction of five-dimensional vector multiplets. Naively the $N = -2$ case appears trivial as this would imply $p = 0$ and that the Hesse potential is just the identity. Despite this, it can be seen by substituting for N into the action, that this is possibly more interesting than this suggests. We would therefore like to include this in our analysis and it will become apparent that this model is related to those that contain gravity where the Hesse potential is not a homogenous polynomial but a logarithm, $\mathcal{V} = -\log\sigma$.

By imposing the extremal instanton ansatz $\partial_m\sigma = \pm\partial_m b$ our equation of motion becomes

$$\partial_m(\sigma^N\partial^m\sigma) = 0 \tag{2.31}$$

which is equivalent to

$$\triangle\sigma^{N+1} = 0 \tag{2.32}$$

Consequently, we have found our dual coordinate of σ, which is given by σ^{N+1}. Close to the centre this solution has the form

$$\sigma^{N+1} \sim \frac{1}{r^2} \tag{2.33}$$

hence implying

$$\sigma \sim r^{\frac{-2}{N+1}}. \tag{2.34}$$

It follows that

$$\sigma \xrightarrow[r\to 0]{} \begin{cases} 0 & \text{if } N < -1 \\ \infty & \text{if } N > -1 \end{cases} \tag{2.35}$$

This behaviour has important significance that we will discuss later.

2.3.2 General homogeneous Hesse potentials

The most simplistic generalisation that we can then apply is to look at Hesse potentials which depend on an arbitrary number of scalar fields and are homogeneous of degree p. In this case the dual scalars

$$\sigma_I \simeq \mathcal{V}_I = \frac{\partial \mathcal{V}}{\partial \sigma^I} \tag{2.36}$$

are homogeneous functions of degree $p-1$ of the scalars σ^I. This also defines our notation where we will be using a subscript capital middle alphabet roman character upon the Hesse potential $\mathcal{V}(\sigma)$ to denote a derivative with respect to the scalar field. It follows from $\triangle \sigma_I = 0$ that the dual scalars have the asymptotics $\sigma_I \sim r^{-2}$ at the centres, implying that

$$\sigma^I \sim r^{\frac{-2}{p-1}}. \tag{2.37}$$

An important question we can investigate is how these solutions behave, particularly at their centres. The scalar fields always run off to 0 or infinity at the centres and hence one can ask if these points are at finite or infinite distance, where we are replacing the concept of distance with a concept of an affine curve parameter. This replacement is essential as it is both sufficient and most simple if the case we can consider is that of a single centred solution where the scalar fields are known to only vary upon an isotropic submanifold. Hence we have to investigate whether $r = 0$ is at a finite or infinitely valued location of an affine parameter along the null geodesic of the corresponding solution. Using the definition of the dual scalars $\triangle \sigma_I = 0$, which is a harmonic map from space-time E to a flat submanifold $N \in M$, and with the radial coordinate, r, as a curve parameter

$$\triangle \sigma_I = \frac{\partial^2 \sigma_I}{\partial r^2} + \frac{3}{r}\frac{\partial \sigma_I}{\partial r} = 0 \tag{2.38}$$

we can introduce the coordinate

$$\tau = \frac{A}{r^2} + B \tag{2.39}$$

where $A \neq 0$, and B are constants and τ is unique up to affine transformations. Hence τ is an affine curve parameter such that

$$\frac{\partial^2}{\partial \tau^2}\sigma_I = 0. \tag{2.40}$$

This is a reparameterisation of the geodesic such that our solution takes the very simple form

$$\sigma_I(\tau) = q_I \tau + \sigma_I(0). \tag{2.41}$$

By using these affine coordinates we see that

$$\lim \tau(r) \xrightarrow[r \to 0]{} \infty \tag{2.42}$$

and this holds irrespective of our choice of affine coordinate. Thus our scalars always run to infinite distance on the scalar manifold, which is different to the behaviour previously observed for solutions such as extremal black holes. In these cases the scalar fields display fixed point behaviour by approaching points that are determined by the charges through black hole attractor equations. We would like to see similar fixed point behaviour for our instanton solutions given a general homogenous potential. However this can be observed if instead we consider ratios of the scalar fields, rather than the individual scalars themselves. Hence at the centres the limits are now finite and depend solely upon the charges

$$\frac{\sigma_I}{\sigma_J} \to \frac{q_I}{q_J} \tag{2.43}$$

An alternative approach would be to perform a (singular) rescaling of our scalar fields such that the new scalar fields are homogenous of degree zero. The natural way of achieving this would be to take the appropriate power of the Hesse potential

$$\tilde{\sigma_I} = \frac{\sigma_I}{\mathcal{V}(\sigma)^{(p-1)/p}} \underset{r\to 0}{\longrightarrow} finite. \tag{2.44}$$

Hence for

$$\sigma_I \sim \frac{1}{r^2}, \sigma^I \left(\frac{1}{r^2}\right)^{1/p-1} \tag{2.45}$$

we would have

$$\mathcal{V}(\sigma) \sim \left(\frac{1}{r^2}\right)^{p/(p-1)}, \mathcal{V}(\sigma)^{(p-1)/p} \sim \frac{1}{r^2}. \tag{2.46}$$

Thus our rescaled $\tilde{\sigma_I}$ would have a finite parameter value. This discussion has given a visual interpretation of the solutions we have generated, however at this level they have no physical meaning. However for logarithmic models and when we couple to gravity this will have physical implications which we will be discussed in chapter 5.

2.3.3 Hesse Potential $\mathcal{V} = \frac{1}{6}C_{IJK}\sigma^I\sigma^J\sigma^K$

It is worthwhile to consider a couple of specific Hesse potentials that are connected to models covered elsewhere in literature. In the case of the temporal reduction of rigid supersymmetric five dimensional vector multiplets, then the most general Hesse potential is the cubic polynomial [28]. We can ignore terms of quadratic order or less, as they either do not contribute (linear and constant terms) or provide only a constant contribution (quadratic terms) to the scalar metric. We therefore look at the cubic polynomial

$$\mathcal{V} = \frac{1}{6}C_{IJK}\sigma^I\sigma^J\sigma^K \tag{2.47}$$

with corresponding metric

$$N_{IJ} = \mathcal{V}_{IJ} = C_{IJK}\sigma^K \tag{2.48}$$

where in the notation we are using $\mathcal{V}_{IJ}$ is the second derivative of the Hesse potential with respect to the σ^I scalar fields. The dual coordinates σ_I are normalised such that

$$\sigma_I = \frac{1}{3}\mathcal{V}_I = \frac{1}{6}N_{IJ}\sigma^K \tag{2.49}$$

and hence

$$\sigma_I\sigma^I = \mathcal{V}(\sigma) \tag{2.50}$$

We obtain the multi-centred solution

$$\sigma_I = h_I + \sum_{a=1}^{n} \frac{q_{Ia}}{|x - x_a|^2}. \tag{2.51}$$

It is generally not possible to find an explicit solution for σ^I in terms of σ_I in terms of the harmonic functions. However this is possible if we look at a specific case that is related to the STU-model. In this case we have the more specific Hesse potential

$$\mathcal{V} = \sigma^1\sigma^2\sigma^3 \tag{2.52}$$

Renormalising our scalar fields, we can then write our dual fields as

$$\sigma_1 = \sigma^2\sigma^3, \sigma_2 = \sigma^3\sigma^1, \sigma_3 = \sigma^1\sigma^2 \tag{2.53}$$

and hence our solution is

$$\sigma_I = H_I \tag{2.54}$$

where H_I, $I = 1, 2, 3$ are harmonic functions. We now obtain the explicit solutions

$$\sigma^1 = \sqrt{\frac{\sigma_2\sigma_3}{\sigma_1}} = \sqrt{\frac{H_2H_3}{H_1}}, \tag{2.55}$$

with cyclic permutations of $1, 2, 3$ for the other solutions. It is now explicitly clear that the fields σ^I diverge like $\frac{1}{r}$ for $r \to 0$ whilst the ratios are finite and depend only upon charges

$$\frac{\sigma^1}{\sigma^2} = \frac{H_2}{H_1} \to \frac{q_2}{q_1}. \tag{2.56}$$

2.3.4 Hesse Potential $\mathcal{V} = \frac{1}{4!}C_{IJKL}\sigma^I\sigma^J\sigma^K\sigma^L$

We can now extend our analysis to potentials of higher order, in this case a quartic Hesse potential. Whilst this is not extendable to a 5-dimensional supersymmetric model, which requires at most a cubic Hesse potential to ensure gauge symmetry, it is still interesting to see that solutions can be constructed for such a model. The corresponding metric is

$$N_{IJ} = \frac{1}{2}C_{IJKL}\sigma^K\sigma^L \tag{2.57}$$

and the dual coordinates are given by

$$\sigma_I = \frac{1}{4!} C_{IJKL} \sigma^J \sigma^K \sigma^L = \frac{1}{6} \mathcal{V}_I \tag{2.58}$$

Similarly to the previous examples these dual coordinates solve the harmonic equation of motion, and hence we can be given as harmonic functions $\sigma_I = H_I$. Generally an explicit solution for σ^I is not possible, however homogeneity implies that $\sigma^U \sim r^{-2/3}$ as $r \to 0$ and hence the ratios of the scalars have finite limits.

Also similarly to the previous examples, it is possible to find explicit solutions for simple choices of Hesse potentials. If we take a similar form of potential to the STU connected model, however quartic instead of cubic

$$\mathcal{V} = \sigma^1 \sigma^2 \sigma^3 \sigma^4 \tag{2.59}$$

we can then normalise the dual fields ($\sigma_1 = \sigma^2 \sigma^3 \sigma^4$, etc.), and find the solutions

$$\sigma^1 = \left(\frac{\sigma_2 \sigma_3 \sigma_3}{(\sigma_1)^2} \right)^{1/3} = \left(\frac{H_2 H_3 H_4}{H_1} \right)^{1/3} \tag{2.60}$$

with cyclic permutations for σ^2, σ^3 and σ^4.

2.3.5 Hesse Potential $\mathcal{V} = -\log(\sigma)$

Logarithmic Hesse potentials are important solutions for our analysis as they are the first set of solutions that we can lift to 5-dimensional Einstein-Maxwell type theories. The simplest theory that we can start with is that of a single scalar

$$\mathcal{V} = -\log \sigma \tag{2.61}$$

where $\sigma > 0$. The resulting metric is

$$\mathcal{V}'' = \frac{1}{\sigma^2} \tag{2.62}$$

This model is of the same class as the first model we investigated, and hence has similar properties. The dual coordinate is proportional to $\mathcal{V}'$ and can be normalised to

$$\sigma' = \frac{1}{\sigma} \tag{2.63}$$

hence we have our equation of motion $\triangle \sigma' = 0$ with solutions

$$\sigma = \frac{1}{H}. \tag{2.64}$$

We can see the behaviour of the solution by considering the single centred solution

$$\sigma = \frac{1}{h + \frac{q}{r^2}}. \tag{2.65}$$

It follows that

$$\sigma \xrightarrow[r \to \infty]{} \frac{1}{h}, \sigma \xrightarrow[r \to 0]{} 0. \tag{2.66}$$

2.3.6 Hesse Potential $\mathcal{V} = -\log(\sigma^1\sigma^2\sigma^3)$

The next interesting step that we can take is to add additional fields to the Hesse potential. The Euclidean version of the STU-model is obtained through considering

$$\mathcal{V} = -\log(\sigma^1\sigma^2\sigma^3) = -\log\sigma^1 - log\sigma^2 - log\sigma^3. \tag{2.67}$$

and hence this model is simply three copies of the previous one. Hence we have similar solutions to those we determined in the last example, with dual coordinates

$$\sigma_I = \frac{1}{\sigma^I} \tag{2.68}$$

and explicit solutions can be found

$$\sigma^I = \frac{1}{H_I} \tag{2.69}$$

We will discuss later the geometry of this solution, and how this can be then lifted to a five dimensional extremal black hole solution of five dimensional supergravity.

2.3.7 Hesse Potential $\mathcal{V} = -\log\hat{\mathcal{V}}(\sigma)$ with homogeneous $\hat{\mathcal{V}}(\sigma)$

The final generalisation we can consider is to look at logarithmic potentials of homogenous functions of arbitrary degree

$$\mathcal{V}(\sigma^I) = -\log\hat{\mathcal{V}}(\sigma^I) \tag{2.70}$$

where

$$\hat{\mathcal{V}}(\lambda\sigma^I) = \lambda^p\hat{\mathcal{V}}(\sigma^I) \tag{2.71}$$

where p is an integer. Hence the Hesse potential is not strictly a homogeneous function but is homogenous of degree zero up to a constant shift. Thus the first derivatives

$$\sigma_I \simeq \frac{\partial\mathcal{V}}{\partial\sigma^I} \tag{2.72}$$

are homogeneous of degree -1, and the metric

$$N_{IJ} = \frac{\partial^2\mathcal{V}}{\partial\sigma^I\partial\sigma^J} \tag{2.73}$$

is homogeneous of degree -2. We therefore observe solutions that correspond to the case N=-2 in our first two examples in this section. In particular we see that the solutions show the fixed point behaviour where they run off to infinite distance of an affine parameter, whereas the ratios approach finite values determined by the charges.

2.4 Dimensional lifting without gravity

One of the main aims of this work is to investigate the connection between solution within five-dimensional theories and the four-dimensional instantons that we have calculated. We are therefore interested in seeing how our models lift to five dimensions. Initially we will look at how the theories lift without including gravity and hence obtain charged soliton solutions. The five dimensional theory we can obtain from lifting our 4-dimensional action (2.1) has the form

$$S[\sigma, A_\mu] = \int d^5x \left(-\frac{1}{2} N_{IJ}(\sigma)\partial_\mu \sigma^I \partial^\mu \sigma^J - \frac{1}{4} N_{IJ}(\sigma) F^I_{\mu\nu} F^{\mu\nu|J} + \ldots \right) \tag{2.74}$$

where $\mu, \nu = 0, 1, 2, 3, 4$ are five-dimensional Lorentz indices and the four dimensional axionic field b^I has been lifted to the time components of a five-dimensional gauge field

$$b^I = -A^I_0 \tag{2.75}$$

To ensure the theory is covariant, we also include the magnetic components of the five-dimensional field strength, and allow for additional contributions provided they do not contribute to the four dimensional model when we perform a temporal reduction. Hence by restricting to static and purely electric configurations, and reducing with respect to time we regain our original action. The instanton solutions are therefore associated with electrically charged solitons. The five-dimensional theory has the equations of motion

$$N_{KJ}\Box\sigma^J + \frac{1}{2}\partial_K N^{IJ}\partial_\mu \sigma^I \partial^\mu \sigma^J = \frac{1}{4}\partial_K N_{IJ} F^I_{\mu\nu} F^{\mu\nu|J} \tag{2.76}$$

$$\partial_\mu \left(N_{IJ} F^{\mu\nu|J} \right) = 0 \tag{2.77}$$

As we are interested in the properties of our instanton solution we restrict this to background where the solution is static and does not carry magnetic charge. This ensures all time derivatives vanish, and the only non-vanishing field strength components can be expressed in terms of the electrostatic potentials A^I_0, such that $F_{tm} = -F_{mt} = -\partial_m A^I_0 = \partial_m b^I$ and our hence the equations of motion take the form

$$N_{KJ}\triangle\sigma^J + \frac{1}{2}\partial_K N_{IJ}\partial_m \sigma^I \partial^m \sigma^J = \frac{1}{2}\partial_K N_{IJ} F^I_{0m} F^{0m|J} \tag{2.78}$$

$$\partial_m \left(N_{IJ}\partial^m A^J_0 \right) = 0 \tag{2.79}$$

By substituting in our axionic scalar, b^I, these equations of motion are identical to those we obtained for our four dimensional theory. It follows that our extremal instanton ansatz (2.13) corresponds to imposing

$$\partial_m \sigma^I = \pm F^I_{0m} \tag{2.80}$$

and hence that the scalars σ^I are proportional to electrostatic potentials. For the five dimensional vector multiplets this is the condition for a BPS solution with scalars and electric fields. We can use these results to compare how the properties of our five-dimensional electrically charged soliton relate to our four dimensional instantons

2.5 Para-holomorphic Coordinates and Para-Kählar Manifolds

Before we investigate the properties of our solutions it is worth investigating the geometry underlying the models that we have developed within this thesis. For metrics with a target space $\tilde{M}$ of the form

$$d\tilde{s}^2 = N_{IJ}\left(d\sigma^I d\sigma^J + db^I db^J\right) \tag{2.81}$$

it is possible to introduce complex coordinates to obtain a Hermitian line element

$$d\tilde{s}^2 = N_{IJ} dX^I dX^J \tag{2.82}$$

However for the action (2.1) with target space M which we have focused upon throughout this thesis, we have the form

$$ds^2 = N_{IJ}(\sigma)\left(d\sigma^I d\sigma^J - db^I db^J\right). \tag{2.83}$$

which differs from (2.81) by a relative sign.

This geometry can be investigated by introducing the concept of a para-holomorphic coordinate. This involves retaining an analogous structure to the imaginary number i whilst replacing the imaginary unit with the para-complex unit e. Whilst i is defined by $i^2 = -1$ and $\bar{i} = -i$, the para-complex unit is defined as $e^2 = 1$ and $\bar{e} = -e$.

The structure of the geometry of para-complex numbers was investigated in detail in the review by Crunceanu et al. [45] and we will discuss their key features relevant for this thesis here. Furthermore it has already been shown that para-complex numbers can be used to derive a Euclidean version of supersymmetry within the original paper on the supergravity description of the D-instanton [5]. For more detailed information upon the para-complex geometry of rigid and local Euclidean vector multiplets we refer the author to [30, 17] and for recent mathematical discussions to [46].

For the work within this thesis, we can introduce the para-holomorphic coordinates

$$X^I = \sigma^I + e b^I \tag{2.84}$$

such that the line element becomes manifestly para-hermitian

$$ds^2_M = N_{IJ} dX^I d\overline{X}^J. \tag{2.85}$$

Analogous to the approach in the theory of complex manifolds we define an almost para-complex structure on a $2n$-dimensional manifold as a tensor field J_n^m of type (1,1) where

$$J_n^m J_p^n = \delta_p^m \tag{2.86}$$

with the additional condition that there are an equal number of eigenvalues of 1 and -1. The almost para-complex structure is called integrable if the Nienhuis tensor vanishes. The J tensor in terms of para-holomorphic coordinates is then diagonal with eigenvalues $\pm e$. An integrable almost para-complex structure is then simply referred to as a para-complex structure.

If the manifold also carries a metric then a compatibility condition between the metric and the para-complex structure can be imposed. The metric g_{mn} is then para-hermitian if it has a para-complex structure which is also an anti-isometry

$$g_{mn} J_p^m J_q^n = -g_{pq}. \tag{2.87}$$

This is equivalent to imposing that the metric has only mixed para-holomorphic and anti-para-holomorphic indices so that there the line element is of the form

$$g_{I\overline{J}} dX^I d\overline{X}^J. \tag{2.88}$$

It follows that a fundamental antisymmetric two-form

$$\omega_{mn} := g_{mp} I_n^P \tag{2.89}$$

can be defined. Where this two form is closed this further allows one to define para-Kähler manifolds as para-hermitian. It follows that the metric can be defined in terms of a para-Kähler potential

$$g_{I\overline{J}}(X, \overline{X}) = \frac{\partial^2 K(X, \overline{X})}{\partial X^I \partial \overline{X}^J} \tag{2.90}$$

It is then also possible to define special para-Kähler manifolds where the Kähler potential is expressible in terms of a para-holomorphic prepotential

$$K(X, \bar{X}) = e \begin{pmatrix} \bar{X}^I & \bar{F}_I(\bar{X}) \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} X^I \\ F_I(X) \end{pmatrix}. \tag{2.91}$$

These manifolds are called affine special para-Kähler manifolds and are the target spaces for rigid Euclidean vector multiplets.

A feature of the dimensional reduction over time is that it generates such para-Kähler manifolds from Hessian manifolds. Considering dimensional reduction without coupling to gravity the line element

$$ds^2 = N_{IJ} d\sigma^I d\sigma^J \tag{2.92}$$

maps

$$ds^2_M = N_{IJ}\left(d\sigma^I d\sigma^J - db^I db^J\right) = N_{IJ} dX^I d\overline{X}^J \tag{2.93}$$

such that if the original line element is Hessian then it follows that the new one is para-Kähler

$$N_{IJ}(\sigma) = \frac{\partial \mathcal{V}}{\partial\sigma^I \partial\sigma^J} \Rightarrow N_{IJ}(X + \overline{X}) = \frac{\mathcal{K}}{\partial X^I \partial \overline{X}^J} \tag{2.94}$$

where

$$K(X + \overline{X}) = 4\mathcal{V}(\sigma(X, \overline{X})). \tag{2.95}$$

We can now rewrite our Euclidean four dimensional action in terms of these para-holomorphic coordinates

$$S[X]_{(0,4)} = \int d^4x \frac{1}{2} N_{IJ}(X + \overline{X}) \partial_m X^I \partial^m \overline{X}^J \tag{2.96}$$

where the metric is defined by a para-Kähler prepotential which corresponds to the Hesse potential which was discussed in the previous section.

2.5.1 Hesse Potentials and Para-Kähler Potentials

The logarithmic Hesse potential that we evaluated in section 2.3.5 provides the action

$$S = \int^4 x \frac{1}{\sigma^2} \left(\partial_m \partial^m \sigma - \partial_m b \partial^m b\right) \tag{2.97}$$

which can be rewritten in terms of para-holomorphic coordinates as

$$S = \int d^4x \frac{\partial_m X \partial^m \bar{X}}{(\mathbb{R}(X))^2} \tag{2.98}$$

and hence this is a model with a para-Kähler target space with para-Kähler potential $K = -\log(X + \bar{X})$. The target space of this model is the symmetric space $SL(2,\mathbb{R})/SO(1,1)$ which is AdS^2,

As before, this can then be extended to include further sigma fields, which due to the behaviour of the logarithm under products of the sigma fields provides multiple copies of this model. With 3 such copies then the STU model with para-Kähler potential

$$K = -\log((X^1 + \bar{X}^1)(X^2 + \bar{X}^2)(X^3 + \bar{X}^3)) \tag{2.99}$$

is generated. The target space of this model is even projective special para-Kähler and is covered in detail in [17]. However the para-holomorphic prepotential can then be seen to be given by

$$F = -\frac{X^1 X^2 X^3}{X^0} \tag{2.100}$$

which is the form expected for Euclidean vector multiplets coupled to gravity. However before we extend our model to include gravity we would now like to investigate the properties of our instanton solutions.

2.6 Summary

By introducing a Wick rotated four dimensional non-linear sigma model which can be constructed through the dimensional reduction of a five dimensional theory which is not coupled to gravity we then discussed the conditions which allow for us to construct instanton type solutions. We then investigated the equations of motion of the model and introduced the instanton ansatz (2.13) which allowed us to construct multi-centred solutions without requiring supersymmetry. This was achieved by requiring the metric to be defined to be the second derivative of a Hesse potential. A number of explicit examples were then discussed.

The solutions which have been generated are expected to correspond to instanton solutions that can be lifted to black holes in a higher dimensional theory and this is developed further in chapter 5. However this still leaves a number of questions unanswered. Firstly, are these solutions true instanton solutions and if so, how have we managed to circumvent Derricks Theorem? Through investigating the instanton amplitudes this discussed in detail in chapter 4. Before we discuss this, we first would like to understand the properties of our solution, their geometry and how they correspond to the supersymmetric models which have been discussed in detail in the literature.

Chapter 3

Properties of Instanton Solutions

The instanton solutions that we have generated from our model have a number of associated properties. They carry charge, and have a particular geometrical interpretation. The models can also be understood in terms of Hodge dual tensor fields, and through investigating this aspect we will motivate the instanton action which allows us to understand the behaviour of the instanton solution from the perspective of the behaviour at a boundary. Finally the models we have currently looked at have not required any discussion of supersymmetry as our requirement for them to originate from models defined with a Hessian potential allows us to discuss a more general set of models which include those of supersymmetry as a subset. We will conclude this section by discussing how our instanton solutions are related to the supersymmetric case. The major emphasis is on work published in [47] however, aspects of the discussion, particular the first two sections are based upon work in [41].

3.1 Instanton charges

3.1.1 Harmonic functions and instanton charges

By investigating the connection between harmonic functions and the existence of a reduction of the second order equations of motion to first order, we now would like to justify our assertion that the parameters q_I in our Harmonic functions can be interpreted as charges, and hence also present a definition of instanton charge. This will allow us to derive our instanton solutions from a different perspective. It has already been shown that this connection exists for extremal non-BPS black holes [42, 43, 48]. We will show that by imposing that solutions carry finite instanton charges automatically implies that the equations of motion can be replaced by first order equations.

The model which we have presented has already been shown to have a constant shift symmetry of $b^I \to b^I + C^I$ for our target manifold M and hence implies the existence of n charges and an associated current

$$j_I = \partial_m \left(N_{IJ}(\sigma) \partial^m b^J \right) . \tag{3.1}$$

The charge associated with this current is simply

$$Q_I = \int d^4 x j_I = \oint d^3\Sigma^M \left(N_{IJ}(\sigma)\partial_m b^J\right). \tag{3.2}$$

where we have used the fact that J_I is a total derivative in order to rewrite Q_I as a surface charge.

Due to our equation of motion $\partial^m(N_{IJ}\partial_m b^J)$ these charges vanish identically, unless we also include sources. Throughout this thesis we will only consider point-like sources, located where the harmonic functions which parameterise the solution have poles. In Euclidean theories such point-like sources, which are located in space and time, are referred to as (-1) branes and interpreted as instantons. Hence we can extend our theories in the same way as the theories of topics such as string dualities are also extended and which our class of action provide models for.

In order that we have non-vanishing instanton charge we also require particular behaviour at the asymptotics of our solution. By assuming that our solution is restricted to a finite region, we can then take the limit $r \to \infty$, where r is the radial coordinate of the region. Expanding in powers of $\frac{1}{r}$ the contribution of the leading term of the solution should be finite and non-vanishing, while sub-leading terms should not contribute. Taking the integration surface to be S^3_r, with radius r and integrating over the angles which our solution does not depend upon our resultant charge is

$$Q_I = 2\pi^2 \lim_{r\to\infty}\left(r^3 N_{IJ}(\sigma)\partial_r b^J\right) \tag{3.3}$$

where the $2\pi^2$ is the volume of the unit three-sphere. As we are looking for finite and also non trivial solutions, we assume that Q_I is neither infinite or zero. Hence we expect the leading term to be of the form $\frac{1}{r^3}$ and therefore this term is the derivative of a spherically symmetric harmonic function $\tilde{H}_I(r)$

$$N_{IJ}\partial_r b^J = \frac{1}{2\pi^2}\frac{Q_I}{r^3} + \dots = \partial_m \tilde{H}_I(r) + \dots \tag{3.4}$$

with

$$\tilde{H}_I(r) = \frac{\tilde{q}_I}{r^2} + \tilde{h}_I \tag{3.5}$$

In direct comparison to before this leaves us with two distinct cases. The first case requires imposing that the full solution is spherically symmetric. It then follows that a solution is obtained by setting all sub-leading terms to zero and imposing the extremal instanton ansatz (2.13). Hence we observe that the solution is spherically symmetric with the modification of a δ-function type source term at $r = 0$, and magnitude q_I. Hence this is interpreted as a (-1)-brane of total charge $\tilde{q}_I$

Alternatively for non spherically symmetric solutions we have to impose that the asymptotics are subject to the extremal instanton ansatz. Thus we obtain instanton solutions provided

$$N_{IJ}\partial_m b^J = \partial_m \tilde{H}_I(x) \tag{3.6}$$

The right hand side is a total derivative and hence this requires an integrability condition similar to (2.19) and the condition on the scalar metric $N_{IJ} \oplus (-N_{IJ})$ of M to be para-Kähler. It then follows that the second order equations of motion have be reduced to first order quasi-linear partial differential equations

$$\partial_m b^I = N^{IJ} \partial_m \tilde{H}_J(x) \tag{3.7}$$

where N^{IJ} is the inverse of N_{IJ}. This is only possible if there exist charges Q_I which prescribe the asymptotic behavior of the solutions.

Solutions with the correct asymptotics are then given by the standard harmonic functions

$$\tilde{H}_I(x) = \tilde{h}_I + \sum_{a=1}^{N} \frac{\tilde{q}_{aI}}{|x - x_a|^2} \tag{3.8}$$

with $x, x_a \in \mathbb{R}^4$. The leading term is then

$$\tilde{H}_I(x) \approx \frac{1}{|x|^2} \sum_{a=1}^{N} \tilde{q_a I} + \mathcal{O}(|x|^{-3}) \tag{3.9}$$

and hence the total instanton charge is $\tilde{q}_I = \sum_{a=1}^{N} \tilde{q}_a I$

The relation between the single centered solution and the versions calculated in chapter 2 can be seen by applying the generalised extremal instanton ansatz

$$\partial_r \sigma_I = N_{IJ} \partial_r \sigma^I = N_{IJ} R^J_K \partial_r b^K = R^J_I N_{JK} \partial_r b^K \tag{3.10}$$

where R^I_J is a rotation matrix which generalises the instanton ansatz. It follows that the solutions we obtain are equal up to an additive constant

$$\partial_m H_I = R^J_I \partial_m \tilde{H}_J \tag{3.11}$$

However, the coefficients of the harmonic functions are then also related through the rotation matrix

$$q_I = R^J_I \tilde{q}_J \tag{3.12}$$

Thus we have shown q_I does have the form of a charge, and that by making this assertion we can obtain our instanton solutions as we did before within chapter 2. Furthermore, this equation is related to the multi-centred solution as it would refer to the centre of a single centre where all centres would have the same R-matrix.

3.2 Dualisation

The interpretation of our solutions as instantons would suggest that we expect that by substituting into the action we would obtain a finite and positive result that is proportional to the charges. However from our scalar fields only varying along null directions of the target space it is obvious that our action evaluated upon these solutions is identically zero. This is an inherent problem of an action that is not positive definite, and exhibits such solutions. However to interpret them as instanton solutions we require a non trivial instanton action.

This problem has also been encountered in the type IIb D-Instanton solutions [5, 11], and it has been observed that a finite instanton action can be obtained by working with the dual form of the theory. This requires the dualisation of the axionic scalars b^I into tensor fields, such that the sigma model action (2.1) can be rewritten to a tensor form.

3.2.1 The Dual form of the $D = 4$ sigma model

For the sigma model we investigate within this model, the axionic scalars b^I only appear in the field equations through their field strengths $F_m^I = \partial_m b^I$. We re-express these in terms of Hodge dual three-forms $H_{mnp|I}$ which by construction satisfy the Bianchi identities

$$\partial_{[m} H_{npq]|I} = 0. \tag{3.13}$$

These can then be written, at least locally, as the exterior derivatives of two-form gauge fields $H_{mnp|I} = 3!\partial_{[m} B_{np]|I}$ such that the standard lagrangian for a theory of scalars σ^I and a two-form gauge field $B_{mn|I}$ has the form

$$\mathcal{L} = -\frac{1}{2} N_{IJ}(\sigma)\partial_m \sigma^I \partial_m \sigma^J - \frac{1}{2.3!} N^{IJ}(\sigma) H_{mnp|I} H_J^{mnp}. \tag{3.14}$$

The euclidean form is obtained by applying wick rotation and hence the resulting Euclidean action is positive definite

$$S_E[\sigma, B] = -\int d^4x \mathcal{L}_E \tag{3.15}$$

This action is equivalent to the Euclidean sigma model action that we obtain (2.1) in that it has the same equations of motion. We use a Lagrange multiplier to promote the Bianchi identity (3.13) into an equation of motion which follows from the Lagrangian

$$\begin{aligned} S_E &= \int d^4x \Big(\frac{1}{2} N_{IJ}(\sigma)\partial_m \sigma^I \partial_m \sigma^J + \frac{1}{2.3!} N^{IJ}(\sigma) H_{mnp|I} H_J^{mnp} \\ &\quad + \lambda b^I \epsilon^{mnpq} \partial_m H_{npq|J} \Big) \end{aligned} \tag{3.16}$$

where b^I is the Lagrange multiplier, and λ is a normalisation constant.

The variation of the action with respect to the tensor field H_I^{mnp} gives

$$H_I^{mnp} = 3!\lambda N_{IJ}(\sigma)\epsilon^{mnpq}\partial_q b^J \tag{3.17}$$

and hence we see that H_I^{mnp} and $\partial_m b^I$ are Hodge duals. By substituting back into the action and integrating by parts it follows

$$\begin{aligned} S[\sigma, b] &= \int d^4x \Big(\frac{1}{2} N_{IJ}(\sigma)\partial_m \sigma^I \partial^m \sigma^J - \frac{1}{2}(3!\lambda)^2 N_{IJ}\partial_m b^I \partial^m b^J \Big) \\ &\quad + (3!\lambda)^2 \oint d^3\Sigma^m b^I N_{IJ} \partial_m b^J. \end{aligned} \tag{3.18}$$

By setting $(3!\lambda)^2 = 1$ we therefore obtain our bulk action (2.1), with an additional surface boundary term. We can also calculate the variation of σ^I and hence obtain the equations of motion for σ^I and $B_{mn|I}$

$$\partial^m \left(N_{IJ}\partial_m\sigma^J\right) = \frac{1}{2}\partial_I N_{JK}\partial_m\sigma^J\partial^m\sigma^K \tag{3.19}$$

$$\partial^m \left(N^{IJ}H_{mnp|J}\right) = 0 \tag{3.20}$$

By substituting (3.17) it is observed that these are equivalent to our equations of motion (2.9).

Alternatively we could have taken the dual action (3.15) and applied the Bogomol'nyi trick to rewrite it as a sum of perfect squares with a remainder term

$$S[\sigma, B] = \int d^4x \left[\frac{1}{2}\left(\partial_m\sigma^I \mp \frac{1}{3!}N^{IJ}\epsilon_{mnpq}H_J^{npq}\right)^2 \pm \frac{1}{3!}\partial_m\sigma^I\epsilon^{mnpq}H_{npq|I}\right]. \tag{3.21}$$

By equating the bulk term to zero we obtain a minimum of the action. This occurs when

$$\boxed{\partial_m\sigma^I = \pm\frac{1}{3!}N^{IJ}\epsilon_{mnpq}H_J^{npq}} \tag{3.22}$$

which is the Hodge dual version of our extremal instanton ansatz. By imposing this ansatz, as well as the Bianchi identity it follows that we obtain the same equations of motion as we obtained in the original theory. This extremal instanton ansatz is similar to the (anti-)self-duality condition observed for Yang-Mills instantons, which is only easily seen in the dual form.

3.2.2 The Instanton Action

By evaluating the bulk action upon our instanton solution in the scalar theory, we observe that the action trivially vanishes. However in the dual form we observe an additional boundary term which now expect to allow us to construct the action of our instanton solution. By substituting the dual instanton ansatz back into the action we obtain the instanton action

$$S_{inst} = \int d^4x N_{IJ}\partial_m\sigma^I\partial^m\sigma^J \tag{3.23}$$

which can be rewritten as a boundary term up to those terms proportional to the equations of motion

$$S_{inst} = \oint d^3\Sigma^m N_{IJ}\sigma^I\partial_m\sigma^J. \tag{3.24}$$

Similarly to the Yang Mills instantons, we suspect this to be possible to express in terms of charges. As the B-field has an abelian gauge symmetry $B_{mn} \to B_{mn} + 2\partial_{[m}\Lambda_{n]}$ we can define the magnetic charge

$$Q_I = \frac{1}{3!} \oint d^3\Sigma^m \epsilon_{mnpq} H_I^{npq} \tag{3.25}$$

where the normalisation has been chosen to be comparable with the charges we calculated from the scalar theory. Applying the extremal dual instanton ansatz it follows that

$$Q_I = \oint d^3\Sigma^m N_{IJ} \partial_m \sigma^J. \tag{3.26}$$

This can then be compared to the instanton action (3.23), and we see that

$$S_{inst} = \sigma^I(\infty) Q_I \tag{3.27}$$

provided the boundary term from the centres of the harmonic functions do not contribute.

This assumption can be investigated by considering the contribution of a single centre to the instanton action

$$\lim_{r\to 0} \oint_{S_r^3} d^3\Sigma^m N_{IJ} \sigma^I \partial_m \sigma^J = \lim_{r\to 0} 2\pi^2 r^3 N_{IJ} \sigma^I \partial_r \sigma^J. \tag{3.28}$$

We know that $N_{IJ}\partial_m\sigma^J$ is a partial derivative and hence close to the centre

$$N_{IJ}\partial_r\sigma^J \sim \frac{1}{r^3} \tag{3.29}$$

Hence to have a finite contribution to the instanton action we require a finite limit at the centres of our solutions, otherwise we should not interpret them as instantons. In order to obtain (3.27) we also require the stronger condition that the scalars vanish at the centres. This is true for the scalar D-instanton solutions from string theory.

If one now substitutes the solution into the original action with boundary term added, the result is

$$\boxed{S_{inst} = \oint d^3\Sigma^m N_{IJ} b^I \partial_m b^J = b^I(\infty) Q_I} \tag{3.30}$$

and the σ^I fields have now been replaced with the axionic b^I fields. The extremal instanton ansatz $\partial_m\sigma^I = \partial_m b^I$ implies that σ^I and b^I are equal up to a constant, hence $b^I(\infty) = \sigma^I(\infty) + c^I$. Thus the instanton actions obtained from the dual action and from the original action with boundary term added only agree if the integration constants c^I are chosen to be zero. This reflects that while the dual actions are completely equivalent classically, as they lead to the same equations of motion, they are only equivalent 'up to zero modes' as quantum theories.

The existence of the instanton action motivates our assertions and investigations in chapter 2 where we discussed the fixed point behaviour of models with different Hesse potentials, and hence which models exhibit finite instanton actions. It is therefore instructive for us to briefly revisit the behaviour of these solutions.

3.2.3 Hesse Potentials revisited

Now that we have introduced the concept of the instanton action we can return to our Hesse potentials from section 2.3 and provide a more lucid description of the fixed point behaviour. For simplicity we will discuss the Hesse potential of the form

$$\mathcal{V} = \sigma^p \tag{3.31}$$

as our observations can then be easily extended. We previously observed that

$$\sigma \xrightarrow[r\to 0]{} \begin{cases} 0 & \text{if } N < -1 \\ \infty & \text{if } N > -1 \end{cases} \tag{3.32}$$

due to $\sigma \sim r^{\frac{-2}{N+1}}$.

It follows that we can therefore only find a finite action, with the form $S_{\text{Inst}} = \sigma^I(\infty)Q_I$ for either logarithmic prepotentials or those which are homogeneous of negative degrees $p = 2, 3, \ldots$. However for models with $N > -1$, including the $N = 1$ case which corresponds to the temporal-reduction of five-dimensional vector multiplets, the instanton action is infinite due to the contributions from the centres. Thus these models do not contain proper finite action instanton solutions. Whilst the $N = -1$ case cannot be covered by the above analysis due to the form of the σ coordinate. However one does find that $\log\sigma$ is harmonic and hence the limit at the centre is zero or infinite depending upon the sign of the charge.

3.3 Supersymmetric field configurations

One key property of the solutions that we have generated is that they do not depend upon supersymmetry. However, we would like to provide an explicit example of how our instanton solutions are related to supersymmetric version and for this we will concentrate upon the four dimensional $\mathcal{N} = 2$ Euclidean vector multiplets. Using these vector multiplets we would like to show that the instanton ansatz which we impose naturally drops out of the formalism we require. This requires us to investigate the supersymmetric theory written in terms of para-holomorphic coordinates and hence composed of the fields $(X^I, \lambda_+^{Ii}, F^I_{mn|-}, Y^{ijI})$ where X^I is the para-complex scalar $X^I = \sigma^I + eb^I$, λ_+^{Ii} is the positive chiral spinor and $F^I_{mn|-}$ is the anti-dual field strength.

3.3.1 BPS states

In order to obtain our BPS states we require non-trivial scalar fields which are invariant under half of the Supersymmetry transformations. We can therefore set the fermions and the gauge fields to zero. Using the supersymmetry transformations from the introduction section (1.59) and discussed in detail in [47] we can now write down the following conditions upon our fields:

$$\begin{aligned} \delta\lambda_{+}^{iI} &= -\frac{i}{2}\partial\!\!\!/ X^{I}\epsilon_{-}^{i} - Y^{ijI}\epsilon_{+j} = 0 \\ \delta\lambda_{-}^{iI} &= -\frac{i}{2}\partial\!\!\!/ X^{I}\epsilon_{+}^{i} - Y^{ijI}\epsilon_{-j} = 0 \\ \delta Y^{ijI} &= 0. \end{aligned} \tag{3.33}$$

We can also consistently set $Y^{ijI} = 0$ which is the equation of motion of the auxiliary field in a bosonic background. The remaining scalar action then has the form

$$S = \int d^4x \frac{1}{2} N_{IJ}\left(X, \bar{X}\right) \partial_m X^I \partial^m \bar{X}^J \tag{3.34}$$

with $X^I = \sigma + eb^i$

Hence the remaining condition on the scalar fields is simply

$$-\frac{i}{2}\partial\!\!\!/ X^I \epsilon_{-}^{i} = 0 \tag{3.35}$$

plus its para-complex conjugate. The supersymmetry transformation parameters ϵ^i are symplectic Majorana spinors and hence have been decomposed into para-complex chiral components

$$\epsilon^i = \epsilon_{+}^{i} + \epsilon_{-}^{i} \tag{3.36}$$

such that

$$\epsilon_{\pm}^{i} = \frac{1}{2}\left(\epsilon^i \pm e(-i)\gamma_0\epsilon^i\right). \tag{3.37}$$

We can make the choice that

$$\gamma_0\epsilon^i = -i\epsilon^i \tag{3.38}$$

or that

$$\gamma_0\epsilon^i = i\epsilon^i \tag{3.39}$$

and hence reduce the number of supersymmetry parameters from 8 to 4, and thus the remaining invariant field configurations have 4 Killing spinors and consequently are $\frac{1}{2}$-BPS states. For definiteness we choose the constraint (3.38) and hence

$$\epsilon_{\mp}^{I} = \frac{1}{2}\left(1 \mp e\right)\epsilon^i \tag{3.40}$$

Thus we can substitute back into the constraint on the scalar field, which becomes

$$\partial\!\!\!/(\sigma^I \pm eb^I)(1 \mp e)\epsilon^i = 0 \tag{3.41}$$

We now need to find a non-trivial solution to this para-complex equation. This is one that does not require constant scalar fields and can be found provided we make use of the para-complex relation

$$(1+e)(1-e) = 0 \tag{3.42}$$

Thus we require the first factor to be proportional to $(1 \pm e)$ which occurs when

$$\partial\!\!\!/\sigma^I = \partial\!\!\!/ b^I \tag{3.43}$$

As the γ matrices are linearly independent it follows that we obtain

$$\partial_m \sigma^I = \pm \partial_m b^I \tag{3.44}$$

which is the extremal instanton ansatz (2.13). Therefore $\frac{1}{2}$-BPS states naturally require the extremal instanton ansatz to be imposed in order for non trivial values of the scalar field to be found.

3.3.2 Adapted Coordinates

It is also possible for us to use adapted coordinates rather than para-complex coordinates to explain the behaviour of our scalar fields. These adapted coordinates are defined by

$$X^I_\pm = \sigma^I \pm b^I \tag{3.45}$$

and can be used to understand the behaviour of the fields without requiring the inclusion of the less intuitive para-complex number e. Using these coordinates the Euclidean BPS condition for a purely scalar background is then

$$\begin{aligned} \delta\xi^{iI}_+ &= -\frac{i}{2}\partial\!\!\!/ X^I_+ \eta^i_- \\ \delta\xi^{iI}_- &= -\frac{i}{2}\partial\!\!\!/ X^I_- \eta^i_+ \end{aligned} \tag{3.46}$$

with chiral projections $\xi^{iI}_\pm = \frac{1}{2}(\mathbb{1} \pm (-i)\gamma^0)\xi^{iI}$ of the fermions and $\eta^i_\pm = \frac{1}{2}(\mathbb{1} \pm (-i)\gamma^0)\epsilon^i$ for the supersymmetry parameters.

Now by imposing that $i\gamma^0\epsilon^i = \epsilon^i$ it follows that $\eta^i_+ = 0$ and for the scalar fields

$$\partial\!\!\!/ X^I_+ = 0 \Leftrightarrow \partial_m \sigma^I = -\partial_m b^I. \tag{3.47}$$

Alternatively by imposing that $i\gamma^0\epsilon^i = -\epsilon^i$ then it follows that $\eta^i_+ = 0$ and our condition becomes

$$\partial\!\!\!/ X^I_- = 0 \Leftrightarrow \partial_m \sigma^I = \partial_m b^I \tag{3.48}$$

Thus by combining both cases we recover the extremal instanton ansatz. Throughout this thesis, we could have used adapted coordinates, however we shall favour the use of para-holomorphic coordinates as they provide a more intuitive comparison to the complex geometry one would see in Minkowski solutions.

3.4 Summary

Within this chapter we have investigated the properties of the instanton solutions which we constructed within chapter 2. We have shown that we can associate the parameters Q_I as charges which originate from the shift symmetry of the axionic b^I fields. This allowed us to determine a dualised form of our model where we replaced the axionic fields with antisymmetric tensor fields. Furthermore we show that this generates a positive definite action which saturates a Bogomol'nyi bound. Through substituting our instanton solution back into the action, this is shown to vanish which would suggest that our solutions are not consistent instanton solutions. However through the addition of the boundary term from the dualisation procedure we acquire the result we expect. The next chapter will investigate this boundary term through the evaluation of the instanton transition amplitude.

As these instanton solutions saturate a Bogomol'nyi bound then for the case of supersymmetric models these are then expected to be BPS solutions. We showed that these are then invariant under half of the Euclidean supersymmetry transformations. This is explicitly shown through the para-holomorphic parameterisation before we briefly discuss an alternative description using adapted coordinates.

Chapter 4

Instanton amplitudes

Instantons are important objects in quantum mechanics as they provide non-perturbative corrections to the Euclidean functional integrals evaluated for amplitude calculations. These corrections come from the additional saddle points that such instantons describe and within this chapter we would like to discuss their role in such calculations. We follow the calculations that Mohaupt and I discuss in [47] in the evaluation of the transition amplitude between sectors of different axionic charge.

4.1 A Toy Example

It is informative to start by presenting a simple one-dimensional toy example where we have the integral I over the real line

$$I = \int_{-\infty}^{\infty} dx e^{-f(x)}. \tag{4.1}$$

We identify this as a toy one dimensional equivalent to the partition function

$$Z = \int Db e^{-S_E[b]} \tag{4.2}$$

where we can assume $f(x)$ can be analytically continued into the complex plane $z = x + iy$ with a sharp saddle point at $z_* = i\alpha$ and $\alpha \in \mathbb{R}$. Thus a saddle point approximation can be obtained by performing a Gaussian integration through the saddle point. For the type of functional integrals we are interested in we can now assume that $f(x)$ has a minimum if we are on a contour parallel to the $\mathbb{R}$-axis

$$\partial_x^2 f(z)_{z=z_*} > 0. \tag{4.3}$$

To obtain the saddle point approximation we now expand f to second order and take an integration contour of $z(x) = x + i\alpha$ with $-\infty < x < \infty$. Hence our integration has the approximate form

$$\begin{aligned} I &\approx \int_{-\infty+i\alpha}^{\infty+i\alpha} dz e^{-f(z_*)-\frac{1}{2}f''(z_*)(z-z_*)^2} = \int_{\infty}^{\infty} dx e^{-f(i\alpha)-\frac{1}{2}\partial_x^2 f(i\alpha)x^2} \\ &\simeq \left(\partial_x^2 f(i\alpha)\right)^{-\frac{1}{2}} e^{-f(i\alpha)}. \end{aligned} \tag{4.4}$$

For our instanton solutions the important term is $f(i\alpha)$ which would correspond to the instanton action we discussed in the previous chapter. To have a consistent saddle point approximation we need that $f(i\alpha) > 0$, so that the value of the action at the saddle point is real and positive, and also that $\partial_x^2 f(i\alpha)$ is positive definite so that the Gaussian integral is damped. We can then use this as a building block for calculating the transition amplitude of our instanton solutions. .

4.2 The Wick rotated scalar action

The form of the of the Euclidean action generated from the Wick rotation of the Minkowski action was discussed in section two. The Wick rotation involves taking the time coordinate, and analytically continuing it into the imaginary plane $x^0 = -it$ and hence the Euclidean action we obtain is

$$S_E[\sigma, b] = \frac{1}{2}\int_E d^4x N_{ij}(\sigma)\left(\partial_m \sigma^I \partial^m \sigma^J + \partial_m b^i \partial^m b^j\right) \tag{4.5}$$

It is worth noting that we have slightly changed our notation from the previous chapter, with lower case indices. This is unique to this chapter, as it allows us to have a more intuitive label for our initial and final states. The Wick rotated action is positive definite, and hence by Derrick's theorem contains no real saddle points. If this action had a real saddle point, it would give rise to a meaningful semiclassical approximation, as the partition function in this form is such that we could generate our instanton solutions. We can use this action to calculate a meaningful semi-classical approximation, as the partition function

$$Z = \int \mathcal{D}\sigma \mathcal{D}b e^{-S_E(\sigma,b)} \tag{4.6}$$

is damped. While the lack of non-trivial real saddle points naively rules out instanton contributions to the amplitude, can now show that this action does have complex saddle points which provides a consistent saddle point approximation.

4.2.1 Instanton amplitudes

We would like to calculate the transition amplitudes from our Euclidean action, as we would like to derive the boundary term that provides us with a non vanishing instanton action. This calculation will follow the classical computations by Coleman and Lee for axionic wormholes [9], and was adapted by Gutperle and Chiodaroli for instantons from the hypermultiplet representation [49].

The amplitude that we calculate is the transition between two different axionic field configurations and hence we treat the σ^i fields as spectators throughout. We denote the initial and final field configuration of the $b^i(x)$ fields as $\chi_I^i(x)$ and $\chi_F^i(x)$ with x a coordinate on Euclidean space $E = \mathbb{R}_t \times \mathbb{R}^3$. Finally we take $t_I \to -\infty$ and $t_F \to \infty$ to find the asymptotic configurations. Denoting the initial and final states as $|I\rangle = |\chi_I\rangle$ and $|F\rangle = |\chi_F\rangle$ the Euclidean transition amplitude between the two states is

$$\mathcal{A} = \langle F|e^{-H(t_F - T_I)}|I\rangle \tag{4.7}$$

for a model with Hamiltonian H. It follows that the functional integral we are interested in is

$$\mathcal{A} = \int_{BC} Dbe^{-S_E[b]} \tag{4.8}$$

where the boundary conditions are determined by the initial and final states.

The transitions that we would like to calculate are between states of different initial and final axionic charge Q_i^I and Q_i^F that correspond to the n conserved charges that we observed in our models previously. We are able to perform such a calculation by inserting suitable projection operators such that we are calculating the transition amplitudes between states of prescribed initial and final charge densities $\rho_i^I(\vec{x})$ and $\rho_i^F(\vec{x})$ where

$$Q_i^{I/F} = \int_{I/F} d^3\vec{x}\rho_i^{I/F}(\vec{x}). \tag{4.9}$$

Such projects are equivalent to the insertion of a functional delta function in the amplitude such that the time-like components of the Noether currents $j_m|i(\vec{x})$ take prescribed values at the initial and final times t_I and t_F. Hence our projection operator has the form

$$P_I|I\rangle = \delta(\rho^I - j_t^I)|\chi_I\rangle \tag{4.10}$$

where we have simplified the notation but understand that $j_t^I = j_{t|i}^I(\vec{x})$ is the Noether current, and we have the delta function

$$\delta\left(\rho^I - j_t^I\right) = \prod_{x \in \mathbb{R}^3_{t=t_I}} \prod_i^n = \delta\left(\rho_i^I(\vec{x}) - j_{t|i}^I(\vec{x})\right) \tag{4.11}$$

In order to apply this projection operator to our functional integral we take the Fourier representation of the functional delta function. This leads to an additional functional integral over auxiliary functions $\gamma_I = (\gamma_I^i(\vec{x}))$ and $\gamma_F = (\gamma_F^i(\vec{x}))$ which exist on the initial and final hypersurfaces

$$P_I|I\rangle = \int_I \mathcal{D}\gamma_I e^{-I\int_I d^3\vec{X}(\rho^I - j_t^I)\gamma_I}|\chi_I\rangle. \tag{4.12}$$

We now simplify the notation by using the letters I and F to indicate an integration over the initial hypersurface, or a functional integral over functions upon those hypersurfaces, and omit the arguments of our functions. We also introduce a short-hand notation such that

$$(\rho^I - j_t^I).\gamma_I = (\rho_i^I(\vec{x}) - j_{t|i}^I(x))\gamma_I^i(\vec{x}) \tag{4.13}$$

Recalling that the Noether charge Q_I generated shift symmetries upon the axionic field $b^i \to b^i + C^i$, we also observe that $j_t^I.\gamma_I$ generates shifts with the parameter $\gamma_I^i(\vec{x})$ and the states such that $|\chi_I\rangle >\to |\chi_I + \gamma_I\rangle$. Thus

$$P_I|I\rangle = \int_I D\gamma_I e^{-i\int_I d^3\vec{x}\rho^I.\gamma_I}|\chi_I + \gamma_I\rangle \tag{4.14}$$

$$P_F|F\rangle = \int_F D\gamma_I e^{-i\int_I d^3\vec{x}\rho^F.\gamma_F}|\chi_F + \gamma_F\rangle \tag{4.15}$$

and the charge density projected amplitude becomes

$$\begin{aligned}\tilde{A} &= \langle|P_F e^{-H(t_F-t_I)}P_I|I\rangle \\ &= \int_F \mathcal{D}\gamma_F \int_I \mathcal{D}\gamma_I \int_{BC} \mathcal{D}b e^{i\int_F d^3\vec{x}\rho^F.\gamma_F} e^{-i\int_I d^3\vec{x}\rho^I.\gamma_I} e^{-S_E[b]}\end{aligned} \tag{4.16}$$

where the boundary conditions are given by

$$b^i(\vec{x}, t_I) = \chi_I^i(\vec{x}) + \gamma_I^i(\vec{x}) \quad , \quad b^i(\vec{x}, t_F) = \chi_F^i(\vec{x}) + \gamma_F^i(\vec{x}). \tag{4.17}$$

In order to evaluate these functional integrals we now combine them into a single integral over the axionic fields $b^i(x)$ without boundary conditions. This can be achieved by setting

$$\gamma_{I/F} = \tilde{\gamma}_{I/F} - \chi_{I/F} \tag{4.18}$$

Substituting in the amplitude we therefore obtain

$$\tilde{\mathcal{A}} = \int_F \mathcal{D}\tilde{\gamma}_F \int_I \mathcal{D}\tilde{\gamma}_I \int_{BC} \mathcal{D}b e^{i\int_F d^3\vec{x}(\tilde{\gamma}_F - \chi_F)\rho^F} e^{-i\int_I d^3\vec{x}(\tilde{\gamma}_I - \chi_I)\rho^I} e^{-S_E[b]} \tag{4.19}$$

with updated boundary conditions

$$b^i(\vec{x}, t_{I/F}) = \tilde{\gamma}_{I/F}^i(\vec{x}). \tag{4.20}$$

The functional integrals over the new parameter $\tilde{\gamma}_{I/F}$ are integration over boundary conditions for the axionic fields $b^i(x)$, and hence we have obtained an integral over $b^i(x)$ without boundary conditions.

$$\tilde{\mathcal{A}} = \left(e^{-i\int_F \chi_F.\rho^F + i\int_I \chi_I.\rho^I}\right) \int \mathcal{D}b e^{i\int_F b^i\rho_i^F - i\int_I b^i\rho^I - i} e^{-S_E[b]}. \tag{4.21}$$

Reducing the integral to this form has the effect of incorporating the behaviour of the position eigenstates with respect to the charge eigenstates within a phase factor which is in front of the functional integral. The integral over the b^i fields is unrestricted however it is still dependant upon the boundary conditions through a boundary term. If we dropped this boundary term we would simply obtain the partition function. This boundary term corresponds to the boundary term that was calculated by considering the Hodge dual theory and hence we expect to relate to our instanton action (3.23). We therefore combine the boundary term into the more elegant expression

$$\Sigma = i\int_I d^3\vec{x} b^i \rho_i^I - i\int_F d^3\vec{x} b^i \rho_i^F = -i\oint_{\partial E} b^i \rho_i \tag{4.22}$$

where ∂E is the boundary with orientation such that we have homology class $[\partial E] = [F] - [I]$. For our spherically symmetric instanton solutions, and other similar ones such as the type IIb D-instanton this boundary is then an asymptotic three sphere $\partial = S^3_{\text{inf}}$.

While these are the boundary terms relevant for the transition amplitudes considered in this chapter, the instanton solutions which lift to single and multi-centered black hole solutions have an asymptotic sphere at infinity $\partial E = S^3_{\text{inf}}$. Whilst they also have further boundary components which arise from cutting out small balls around the points where the harmonic functions become singular we have previously shown that these boundary components do not correspond to the instanton action, if we impose that the instanton action is finite.

We can now summarise the amplitude as

$$\boxed{\tilde{\mathcal{A}}(\chi_I, \chi_F, \rho^I, rho^F) = e^{-i\oint \chi \cdot \rho} \int \mathcal{D}b e^{-S_E[b] - \Sigma[b,\rho]}. \qquad (4.23)}$$

This is an important result that shows we can reduce the amplitude calculation to a finite path integral we can calculate with a multiplicative phase term and hence that our instanton solution provides a contribution to the path integral.

In order to consider the saddle point approximation we need to identify the critical points of the action. However, we need to account for the boundary term and we do this by considering the variations of both the bulk term and boundary term independently.

By varying the boundary term, we clearly see that

$$\delta\Sigma = -i \oint d^3\vec{x} \delta b^i \rho_i \tag{4.24}$$

however we also need to consider any boundary terms that are generated from varying the bulk action due to applying the process of integration by parts.

$$\begin{aligned} \delta S &= \int d^4x N_{ij}(\sigma)\partial_m \delta b^i \partial^m b^j \\ &= \int d^4x \left(N_{ij}(\sigma)\partial^m b^j \delta b^i\right) - \int d^4x \partial_m \left(N_{ij}(\sigma)\partial^m b^j\right) \partial b^i \\ &= \oint d^3\vec{x} n^m N_i j(\sigma)\partial_m b^j \delta b^i - \int d^4x \partial_m \left(N_{ij}(\sigma)\partial^m b^j\right) \partial b^i \end{aligned} \tag{4.25}$$

with n^m the outer normal to the boundary. The second term provides the equation of motion of b^i

$$\partial_m(N_{ij}(\sigma)\partial^m b^j) = 0 \tag{4.26}$$

which comes from the shift symmetry of the axionic scalars. However the first term is a contribution to the variation at the boundary and hence the resulting total boundary variation is

$$(\delta S + \delta\Sigma)_{boundary} = \oint \left(n^m N_{ij}(\sigma)\partial_m b^j \delta^i - i\delta b^i \rho_i\right) \tag{4.27}$$

$$= \int_F d^3\vec{x}\left(N_{ij}(\sigma)\partial_t b^j - i\rho_i^F\right)\delta b^i - \int_I d^3\vec{x}\left(N_{ij}(\sigma)\partial_t b^j - i\rho_i^I\right)\delta b^i$$

Hence for the boundary variation to vanish, this implies

$$\left(N_i j(\sigma)\partial_t b^j\right)_{t=t_{I/F}} = i\rho_i^{I/F}(\vec{x}) \tag{4.28}$$

Potentially this is a confusing statement, because one would expect that both sides of the equation should be real. However with further inspection it becomes clear that $\rho_i^{I/F}(\vec{x})$ must be real because these are the charge densities of the physical states between which the amplitude is computed. This implies that the saddle point solution b^i must obey imaginary boundary conditions. However these boundary conditions are physically meaningful as they correspond to the Wick rotated theory where we have introduced a factor of i to the time component of vectors. Thus, we can rewrite this boundary condition in terms of Minkowski space-time as

$$\left(N_i j(a\sigma)\partial_0 b^j\right)_{I/F} = \rho_i^{I/F} \tag{4.29}$$

4.2.2 The saddle point approximation

We now turn our attention to performing the saddle point approximation so that the integral can be evaluated. In order to find explicit non-trivial saddle points we need to reinstate the σ^i scalar fields and hence we have a full bulk action

$$S_E[\sigma, b] = \frac{1}{2}\int d^4x N_{ij}(\sigma)\left(\partial_m\sigma^i\partial^m\sigma^j + \partial_m b^i\partial^m b^j\right) \tag{4.30}$$

and by calculating the variation on σ^i we have the extremal instanton ansatz

$$\partial_m\sigma^i = \pm i\partial_m b^i \tag{4.31}$$

Thus by working with the definite Euclidean action we obtain an imaginary saddle point solution of the bulk action, b_*^i. This is consistent with the boundary variation, and hence it is this saddle point contribution to the amplitude that we will now calculate. Denoting the saddle point by σ_*^i and $b_*^i = i\beta_*^i$ where $\beta_*^i \in \mathbb{R}$ we have

$$N_{ij}(\sigma)\partial_t\beta^j\big|_{I/F} = \rho_i^{I/F} \tag{4.32}$$

from the boundary conditions. The σ^i fields are treated as classical on-shell background fields which are constrained to obey their bulk equations of motion without setting boundary conditions.

The leading quantum amplitude $\tilde{\mathcal{A}}$ comes from evaluating the action at the saddle point where the bulk action has already been noted to vanish due to the imposing of the extremal instanton ansatz

$$S_E[\sigma_*, b_*] = \frac{1}{2}\int d^4x N_{ij}(\sigma_*)\left(\partial_m\sigma_*^i\partial^m\sigma_*^i + \partial_m b_*^i\partial^m b_*^j\right) = 0. \tag{4.33}$$

However we also need to consider the boundary action where

$$\Sigma[b_*] = -i\oint s^3\vec{x}b_*^i\rho_i = \oint d^3\vec{x}\beta_*^i = \beta_*^i(t_F)Q_i^F - \beta_*^i(t_I)Q_i^I \tag{4.34}$$

thus

$$\tilde{\mathcal{A}} \propto e^{-S_* - \Sigma_*} = e^{-\Sigma_*} = e^{-\beta_*^i(t_F)Q_i^F - \beta_*^i(t_I)Q_i^I} \tag{4.35}$$

It therefore follows that if $\beta_*^i(t_F) = \beta_*^i(t_I) = \beta_0$ then

$$\Sigma_\star = \beta_0(Q_i^F - Q_i^I) \tag{4.36}$$

and the instanton amplitude is proportional to the difference between the initial and final state charges as would be expected for a tunneling amplitude between charged ground states. The transition amplitude now depends upon the values of the axionic fields β_*^I hence breaking the continuous shift symmetry of the axions to a discrete subset of imaginary shifts in β^i or equivalently real shifts in b^i.

$$\beta^i \to \beta^i + 2\pi k \Leftrightarrow b^i + 2\pi k \tag{4.37}$$

where $k \in \mathcal{Z}$ Such breaking of the shift symmetry by the instantons is an expected quantum effect and as a global symmetry it does not provide an inconsistency of the theory.

In order to provide a consistent saddle point approximation we require a damped functional integral, that is one whose fluctuation determinant is positive definite. This is the case provided we perform the functional integration along what we call real directions in b^i-field space. This mean that if we shift the integration variable such that

$$b^u = i\beta_*^i + \tilde{b}^i \tag{4.38}$$

with β_*^i the saddle point solution already discussed and $\tilde{\beta}^i$ the new integration variable we obtain

$$\tilde{\mathcal{A}} \propto e^{-\Sigma_*}\int \mathcal{D}\tilde{b}e^{-\frac{1}{2}\int d^4x N_ij(\sigma_*)\partial^m\tilde{b}^i\partial^m\tilde{b}^j}. \tag{4.39}$$

The integrand has zero linear terms in $\tilde{b}^i$ and hence an expansion around the saddle point is valid. Since N_{ij} is assumed positive definite, and we have a damped fluctuation determinant, this leads to a Gaussian Integral. It follows that the saddle point is well defined, and hence by comparing to our toy model, this does provide the features for the axionic function integral.

4.3 The indefinite scalar Euclidean action

Currently we have only considered the Wick rotated Euclidean action. However in the process of constructing instantons in the second chapter we coupled the Wick rotation with an analytic continuation of the axionic field such that $b^i \to ib^i$. For ease of comparison with the positive definite action we set $\beta^i = ib^i$ and hence we are considering the action of the form

$$\tilde{S}_E[\sigma, \beta] = \frac{1}{2}\int_E d^4x N_{IJ}\left(\partial_m \sigma^i \partial^j - \partial_m \beta^i \partial^m \beta^j\right). \tag{4.40}$$

If we substitute the instanton solution $\{\sigma_*^i, \beta_*^i\}$ back into this action we obtain

$$\tilde{S}_E[\sigma_*, \beta_*] = 0 \tag{4.41}$$

as a consequence of the extremal instanton ansatz (2.13) that we apply in order to obtain the solutions. At first glance by circumventing Derrick's Theorem to obtain our instanton solutions it appears that we have restricted our ability to evaluate the instanton amplitude. It also suggests that the interpretation of our solutions requires reconsidering as we require a positive definite bulk action in order to perform a meaningful saddle-point approximation.

4.3.1 Instanton amplitudes

The question we would like to ask is whether or not we can use the indefinite action to calculate the instanton amplitudes. The main point is that we should consider the Euclidean continuation as a method to compute a fixed physical amplitude of a given fixed theory in Minkowski space. We will now show that it therefore follows that different 'Euclidean versions' are different analytic continuations and hence give the same result for physical quantities. An alternative point of view is to take the Euclidean theory as fundamental, and then it is possible that different Euclidean theories lead to different physics on Minkowski space when under analytic continuation. However we then require that the Euclidean theories have a consistent saddle point approximation. For the class of models considered in this Chapter our calculation show that for both the definite and the indefinite Euclidean scalar action there is only one consistent way to perform in each case, each leading to the same theory in Minkowski space.

The initial and final states of the amplitude which we calculate are the charge or position eigenstates of the original Minkowski field b^i. Thus when using the rotated field $\beta^i = -ib^i$ the boundary conditions are

$$i\beta^i(\vec{x}, t_{I/F}) = \chi^i_{I/F}(\vec{x}) \tag{4.42}$$

Hence it is worth reviewing some of the key points of the previous calculation in terms of the variable β^i. In particular the variation of the bulk action $\tilde{S}_E$ with respect to the rotated axions leads to a real saddle point $\beta^i = \beta^i_*$. However the boundary conditions of the functional integral representing the amplitude are now imaginary valued. Thus the charge projected amplitude has the form

$$\tilde{\mathcal{A}} = e^{-i\oint \chi\cdot\rho} \int \mathcal{D}\beta e^{-\tilde{S}_E[\beta]-\tilde{\Sigma}[\beta[\rho]]} \tag{4.43}$$

The phase factor which is in front of the integral is identical to the positive definite case as the physical eigenstates do not depend on the analytic continuation. However in the saddle point approximation the β^i decompose into two components. A real saddle point solution β^i_* and purely imaginary fluctuation $\tilde{\beta}^i = -i\tilde{b}^i$

$$\beta^i = \beta^i_* + \tilde{\beta}^i = \beta^i_* - i\tilde{b}^i. \tag{4.44}$$

Hence the integration is over purely imaginary configurations. This can be understood as requiring the integration to be taken along the imaginary axis.

The boundary term is

$$\tilde{\Sigma} = \oint d^3\vec{x}\beta^i\rho_i \tag{4.45}$$

and the vanishing of the boundary variation, including the contribution from the bulk action evaluated using integration by parts, implies that

$$\left(N_{ij}(\sigma)\partial_t\beta^j\right)_{I/F} = \rho_i^{I/F} \tag{4.46}$$

which is now consistent with a real saddle point solution. Also the Euclidean Noether current is real due to the Wick rotation of the axionic field and time.

The integration over fluctuations takes the form

$$\int \mathcal{D}\tilde{\beta} e^{\frac{1}{2}\int d^4x N_{ij}\partial_m\tilde{\beta}^i\partial^m\tilde{\beta}^j} \tag{4.47}$$

which is damped for positive definite N_{ij} if and only if the fluctuations are purely imaginary and hence consistent with the boundary conditions of the original functional integral being imaginary.

4.4 The Euclidean scalar-tensor action

Within the introduction three different approaches for finding instanton solutions which avoid the problems from Derrick's Theorem were discussed. These previous two sections have discussed the first two approaches, however the final one of these is to consider the scalar-tensor action. In section 3.2 we saw that by substitution of the instanton solution into the scalar tensor action we obtained

$$S_{inst} = \int d^4x N_{ij}(\sigma)\partial_m\sigma^i\partial^m\sigma^j\Big|_* \tag{4.48}$$

where the σ_*^i are obtained by solving the dual action in terms of harmonic functions for our instantons. Modulo the equations of motion, this action is a boundary term

$$S_{inst} = \oint d^3\vec{x} n^m N_{ij}(\sigma)\sigma^i \partial_m \sigma^j \Big|_* \tag{4.49}$$

and re-expressed in terms of the magnetic charges Q_i where

$$Q_i = \oint d^3\vec{x} n^m N_{ij}(\sigma)\partial_m \sigma^j \Big|_* \tag{4.50}$$

Thus the instanton action can be expressed as the difference of the product of charges and fields on the initial and final hypersurfaces located at t_I and t_F respectively.

$$S_{inst} = \sigma^i(tf)Q_i^F - \sigma^i(t_I)Q_i^I|_* \tag{4.51}$$

Comparing this with the instanton action calculation when using the axionic action in the previous sections

$$\Sigma_* = \beta^i(t_F)Q_i^F - \beta^i(t_I)Q_i^I|_* \tag{4.52}$$

it is easy to observe that the results do not quite agree. The reason for this is that the extremal instanton ansatz which we impose

$$\partial_m \sigma^i = \pm \partial_m \beta^i \tag{4.53}$$

only fixes β^i up to integration constants. For classical physics these integration constants are irrelevant due to the continuous shift symmetry of the axions. However, in the quantum theory this continuous symmetry is broken into a discrete subgroup, and hence this is sensitive to the integration constants $C^i = \sigma^i(t_{I/F}) - \beta^i(t_{I/F})$. Hence, in general, instanton actions based on axions and antisymmetric tensor fields differ by the amount

$$S_{inst} - \Sigma_* = C^i(Q_i^F - Q_i^I) \tag{4.54}$$

Thus imposing that the quantum theories of the axion and the antisymmetric tensor fields are equivalent we must also impose that the C^i vanish modulo the remaining discrete shift symmetries. Hence we make a restriction upon the value of the C^i

$$C^i = 2\pi i k \tag{4.55}$$

where $k \in \mathcal{Z}$. This restricts the classically trivial zero mode of the axions and is understood to derive from the axionic shift symmetry being a global symmetry which is broken by quantum effects. However the tensor field theory has a local gauge symmetry which remains unbroken. By imposing this condition on C^i we ensure that the charge sectors and saddle points of both the axionic and tensor theories match.

The scalar-tensor action itself has already been shown to be possible to rewrite into the form

$$
\begin{aligned}
S[\sigma, b] &= \int d^4x \left(\frac{1}{2} N_{ij}(\sigma) \partial_m \sigma^i \partial^m \sigma^j - \frac{1}{2} (3!\lambda)^2 N_{ij} \partial_m b^i \partial^m b^j \right) \\
&\quad + (3!\lambda)^2 \oint d^3\Sigma^m b^i N_{ij} \partial_m b^j . \qquad (4.56)
\end{aligned}
$$

By setting $(3!\lambda)^2 = 1$ which corresponds to real λ_i we obtain the indefinite version of the axionic action, while setting $(3!\lambda)^2 = -1$, which corresponds to imaginary λ_i lead to the positive definite version. These are then accompanied by the boundary term we obtained in the amplitude calculation. This consistency reflects that the boundary term and also the bulk action are related by an analytic continuation. The boundary term depends upon b^i and hence to obtain our original instanton solutions we have to set

$$
C^i = 0 \qquad (4.57)
$$

and thus imposing that the zero modes of the axions b^i are tied to the zero modes of the scalars σ^i.

We have seen that with $\lambda \in \mathbb{R}$ we preserve the saddle points of the scalar tensor action but map a positive definite action into an indefinite action, while if $\lambda \in \mathbb{I}$ we preserve the positive definiteness of the action, but not its saddle points. However, whilst the instanton solution is no longer a real solution, it is still a complex saddle point. It then follows that we seem to have obtained a dilemma for the classical action. Either we preserve the definiteness of the action, or the its saddle points, but not both. However, we can see from our previous discussion of the quantum theory that this is not a real problem. It is not important that we choose the correct action, however we must choose the correct integration contour such that we obtain consistent quantum amplitudes from the functional integration. Hence our ability to calculate the same physical results from our previous two actions.

4.4.1 The Partition Function

We would finally like to show the quantum equivalence of the scalar tensor action with the scalar axion version. However the quantisation of a gauge theory is slightly more complicated as it requires gauge fixing terms and ghosts. More significantly, the antisymmetric tensor fields provide a reducible gauge theory where gauge fixing terms are required for the ghosts, and the introduction of ghosts for ghosts is also required. This however can be circumvented by integrating over the field strength and imposing a Bianchi identity through a functional delta function. We can also ignore the σ^i fields as they are spectators to the dualisation of the tensor fields. We are therefore interested in calculating the simplified partition function

$$Z = \int \mathcal{D}H \exp\left(-\int d^4x \frac{1}{2.3!} N^{ij} H_{mnp|i} H_j^{mnp}\right) \delta\left(\epsilon^{mnpq} \partial_m H_{npq|k}\right) \tag{4.58}$$

Using the fourier functional integral it follows that

$$Z = \int \mathcal{D}H\mathcal{D}b \exp\left(-\int d^4x \left(\frac{1}{2.3!} N^{ij} H_{mnpZi} H_j^{mnp} - i\mu b^i \epsilon^{mnpq} \partial_m H_{npq|i}\right)\right) \tag{4.59}$$

where μ is a normalisation constant that will be specified later. Integrating by parts we obtain

$$\begin{aligned} Z &= \int \mathcal{D}H\mathcal{D}b \exp\left[\left[\int d^4x \left(-\frac{1}{2.3!} N^i j H_{mnp|i} H_j^{mnp} + i\mu \partial_m b^i \epsilon^{mnpq} H_{npq|i}\right)\right.\right. \\ &\quad \left.-\mu i \int d^4x \partial_m \left(\epsilon^{mnpq} b^i H_{npq|i}\right)\right]. \end{aligned} \tag{4.60}$$

Now using the relation

$$\begin{aligned} (H_{mnp|i} - 3!\mu i N_{ij} \epsilon_{mnpq} \partial^q b^j)^2 &= N^{ij} H_{mnp|i} H_j^{mnp} - 2.3!\mu i H_{mnp|i} \epsilon^m npq \partial_q b^i \\ &\quad -3!(3!\mu)^2 N_{ij} \partial_m b^i \partial^m b^j \end{aligned} \tag{4.61}$$

and defining

$$\tilde{H}_{mnp|i} = H_{mnp|i} - 3!\mu i N_{ij} \epsilon_{mnpq} \partial^q b^j \tag{4.62}$$

we can complete the square and shift to integrating over $\tilde{H}_{mnp|i}$. It follows that

$$\begin{aligned} Z &= \int \mathcal{D}\tilde{H}\mathcal{D}b \exp\left(-\int d^4x Nij \tilde{H}_{mnp|i} \tilde{H}_j^{mnp} - \int d^4x (3!\mu) N_{ij} \partial_m b^i \partial^m b^j \right. \\ &\quad \left. \mu i \oint d^3\vec{x} n^m b^i \epsilon_{mnpq} \tilde{H}_i^{npq} + (3!\mu)^2 \oint d^3\vec{x} n^m N_{ij} b^i \partial_m b^j\right). \end{aligned} \tag{4.63}$$

Thus the integration over $\tilde{H}_{mnp|i}$ decouples from the integration over b^i for all apart from the boundary terms. We are able to ignore this additional term as this vanishes for on shell configurations with $H_{mnp|i}$ and $\partial_q b^i$ being Hodge dual. Hence we have a fully decoupled integration for $\tilde{H}_{mnp|i}$ which provides a multiplicative constant we can ignore in our discussion.

The remaining functional integral for the b^i fields is then

$$Z = \int \mathcal{D}b \exp\left(-\int d^4x (3!\mu) N_{ij} \partial_m b^i \partial^m b^j + (3!\mu)^2 \oint d^3\vec{x} n^m N_{ij} b^i \partial_m b^j\right) \tag{4.64}$$

which is the partition function for the definite Euclidean action with the corresponding boundary term. We can now also set the constant μ such that $(3!\mu)^2 = 1$ and regain the standard normalisation we expect. This also implies that $\mu = \pm\lambda$. It is natural to regard μ as a real constant to ensure the functional integral remains damped through restricting the Euclidean action to be definite. However, the shift in the integration over the H-fields is imaginary and the real saddle points which correspond to our instanton solutions become imaginary saddle points under dualisation. Thus complex values of the axionic field b^i must be considered to match the semiclassical expansions of both versions of the theory. Furthermore the presence of the boundary term breaks the continuous shift symmetry of the bulk action with these boundary terms, and hence to obtain the same saddle point behaviour we have to impose that $C^i = 0$.

4.5 Summary

This chapter has focussed upon calculating the transition amplitudes which are dominated by the solutions we generated within chapter 2. Consequently we show that these structures should be interpreted as instantons, which would appear to contradict Derrick's theorem as discussed within the introduction.

Through taking a Minkowski sigma model and applying a standard Wick rotation we obtained a theory with standard kinetic terms that allow for complex saddle points which do not contradict Derricks theorem. Within this chapter we show that we can calculate transition amplitudes of our model which are dominated by these complex saddle points and investigate the properties of these solutions.

Furthermore we also show that we can use a modified Wick rotation where there is an analytical continuation of the axionic scalars and hence generate an action which is no longer positive definite. Through such a transformation of the action, Derricks theorem no longer applies and the solutions we obtained within chapter 2 are now real solutions of this action. It is then possible to still obtain a consistent saddle point approximation through taking the fluctuations around this point to be imaginary. The resulting functional integral is then related to the previous functional integral by a change of variables. The properties of these instanton solutions then describe the physical theory defined in Minkowski space. In particular the boundary conditions belong to the Minkowski theory Hilbert space and hence are not subject to the analytic continuation. It follows that these two different continuations allow for the computation of the same physical quantities.

The third approach which we discussed was the dual formulation where the model is described in terms of antisymmetric tensor fields. As this has a local gauge symmetry Derrick's theorem no longer applies and hence the instanton solutions correspond to real saddle points of a positive definite action. However, this approach has an additional complication at the quantum level as the axionic fields and the antisymmetric tensor fields are no longer completely equivalent. The instanton action in the axionic case depends upon parameters which have no equivalent in the tensor formulation. It then follows the instanton action in both cases are only equivalent when the asymptotic values of the axions are imposed to equal the asymptotic values of the corresponding non-axionic scalars.

We have taken these formulations and shown how we can then use them to calculate physically meaningful contributions to the transition amplitudes and how these then relate to each other. In particular we have shown that irrespective of the formulation of the action, we can understand our instanton solutions as coming from a boundary action with different integration contours, and hence being present in all of these formulations. It is therefore appropriate for us to consider these solutions as instantons, and the next task is to understand how these behave under dimensional lifting.

Chapter 5

Black Hole Solutions

The previous chapters have shown how we generate instanton solutions for the four-dimensional models, and their associated properties and amplitudes. These solutions can be lifted to five dimensional Einstein-Maxwell theories and by incorporating gravity into our models we would expect the soliton solutions that we have previously discussed to correspond to Black Hole solutions. Incorporating gravity adds additional complications that we will discuss in detail, within the next section. In particular we need to identify the type of solutions which reduce to our action, while we allow for any additional terms that are consistent with our method of reduction.

5.1 Dimensional Lifting and Reduction

5.1.1 Coupling models to gravity

Before we can discuss the form of the Black Hole solutions we lift to, we first need to justify the inclusion of gravity within our models. In order to couple our models to gravity such that our instanton solutions remain solutions we require that the energy momentum tensor vanishes. From our action (2.1) we can obtain the energy momentum tensor through the variation with respect to our background metric

$$T_{mn} = N_{IJ}\left(\partial_m\sigma^I\partial_n\sigma^J - \partial_m b^I\partial_n b^J\right) - \frac{1}{2}\delta_{mn}N_{IJ}\left(\partial_l\sigma^I\sigma^L\sigma^J - \partial_l b^I\partial^l b^J\right) \tag{5.1}$$

Our models exist in $D > 2$ dimensions, hence this implies that for T_{mn} to vanish, this is equivalent to

$$N_{IJ}\left(\partial_m\sigma^I\partial_n\sigma^J - \partial_m b^I\partial_n b^J\right) = 0 \tag{5.2}$$

Hence the scalar fields only vary along null directions of the metric $N_{IJ}\oplus(-N_{IJ})$, as we have previously shown in our construction of the instanton solutions in chapter 2. Alternatively this can also be stated as the scalar fields only varying along eigendirections of the para-complex structure of our model, where the sign of the extremal instanton ansatz (2.13) resulting in an eigenvalue of ±1. Hence, for our instanton solutions the energy momentum tensor vanishes.

This is important as it allows us to couple our models to gravity, without demanding a modification of the equations of motion which are relevant to our instanton solution. Hence provided we have the Hamiltonian Constraint that $T_{mn} = 0$ the solutions found without gravity still apply to sigma models of the form

$$S[g,\sigma,b]_{(0,4)} = \int d^4x\sqrt{g}\frac{1}{2}\left(-R + N_{IJ}\partial_m\sigma^I\partial^m\sigma^J - N_{IJ}\partial_m b^I\partial^m b^J\right). \tag{5.3}$$

as we can consistently solve the Einstein equation with a flat metric space-time metric $g_{mn} = \delta_{mn}$.

Similarly to the soliton solutions that we discussed in the introduction, we can see by demanding this Hamiltonian constraint, we require an indefinite target space metric within this model in order to have instanton solutions that are non trivial. If we had a definite target space metric, the only possible scalar field solutions for $T_{mn} = 0$ are constant.

The extremal instanton ansatz provides a sufficient but not necessary condition for the vanishing of the energy momentum tensor. Provided the metric N_{IJ} is invariant under the transformations

$$N_{IJ} \to N_{KL}R^K_I R^L_J \tag{5.4}$$

where R^I_J is a constant matrix, then the instanton ansatz can be generalised to

$$\sigma^I = R^I_J b^J \tag{5.5}$$

and T_{mn} still vanishes. This can be understood geometrically as this transformation corresponds to an isometry of our initial space $N_{IJ} \oplus (-N_{IJ})$ where

$$\sigma^{\to}\sigma^I \quad , \quad b^I \to R^I_J b^J. \tag{5.6}$$

In the context of extremal black hole solutions within supergravity, the $R^I_J \neq \delta^I_J$ correspond to non-BPS solutions [42, 43]. By flipping the charges of the black holes, such that the R-matrices are diagonal with entries ± 1, such that we obtain an overall ± 1 then we have BPS solutions. This is easy to see geometrically as restricting the fields to vary only along the ± 1 eigendirections of the para-complex structure. This allows us to make an important distinction between BPS and non-BPS extremal solutions in our larger class of models, where we do not demand supersymmetry. In our case, we shall only be discussing BPS solutions.

5.1.2 Kaluza-Klein Reduction

For this thesis we would like to lift the instanton solutions we calculated within the first section, coupled with gravity, to five dimensional BPS black hole solutions. When we lifted in the rigid case, we obtained electromagnetic solitons however when coupled with gravity we have the additional complication in that we have to include a Kaluza-Klein scalar and gauge field. In this case we no longer have flat solutions, however we do obtain solutions that are conformally flat in four-dimensions. This is typical of BPS solutions, and in particular for our lifted instantons, BPS extremal black holes and hence provides the motivation for our use of the term extremal for the instanton ansatz (2.13).

The Kaluza Klein gauge field can consistently be set to zero, which for five dimensional solitons restricts us to static solutions, however the Kaluza Klein scalar has to be incorporated into our four dimensional sigma model. By considering the case of temporally reduced five dimensional supergravity coupled with vector multiplets, such as that discussed in [17], we will look at how we incorporate this scalar and then generalise further examples. It would also be possible for us to spatially reduce the dimensions, however this is equivalent to simply switching the signs within the Lagrangian and hence all results here would hold in that case.

Without gravity the five dimensional vector multiplet contains an equal number of gauge fields and scalar fields. However by incorporating gravity, it obtains an additional gauge field called the graviphoton. The geometry that allows for this multiplet is a version of very special real geometry [50] described by the Hesse potential

$$\mathcal{V} = -\log \hat{\mathcal{V}} \tag{5.7}$$

where the prepotential $\hat{\mathcal{V}}$ is a cubic polynomial. This Hesse potential provides the coupling matrix for the gauge fields, whilst the scalar metric which has one less field is the pullback onto the hypersurface $\hat{\mathcal{V}}$=1. Through dimensional reduction each of the original gauge fields results in an axionic scalar, which when combined with the five dimensional scalars and the Kaluza-Klein scalar reproduce sigma models of the form (5.3).

A further important property of the five dimensional sigma field is that to generate a sigma model of the form, we require the metric of the five dimensional scalar sigma model must be restricted to being homogenous of degree -2 in the scalar fields. We will motivate this later; however in section (2.3.7) we saw that a homogenous potential of degree 2 potential is always obtained for any logarithmic prepotential, and thus we can generalise the very special real geometry of our model to include those of arbitrary degree p.

The dimensional reduction we want to perform is over the time component, and hence we are looking to generate a para-r-map between the target spaces of five dimensional and four dimensional vector multiplets. To distinguish this from the R-map mentioned in for the rigid case, we refer to this as a local para-r-map. This differs also from a spatial reduction map between the target spaces which would be a standard r-map and can be obtained from the analytical continuation of what we present here.

To construct our para-r-map, we introduce $n+1$ scalar fields $h = (h^I) = (h^0, h^1, \ldots, h^n)$ which are affine co-ordinates on a $(n+1)$ dimensional Hessian manifold $\tilde{M}_r$. The Hesse potential for this manifold is then $\mathcal{V}(h) = -\log \hat{\mathcal{V}}(h)$ where the prepotential $\hat{\mathcal{V}}(h)$ is homogeneous of degree p

$$\hat{\mathcal{V}}(\lambda h^0, \ldots, \lambda h^n) = \lambda^p \hat{\mathcal{V}}(h^0, \ldots, h^n). \tag{5.8}$$

The derivative with respect to λ is then

$$\hat{\mathcal{V}}_I(\lambda h) h^I = p\lambda^p - 1\hat{V}(h) \tag{5.9}$$

where we are using notation with a subscript I denoting differentiating with respect to h^I. We can then set $\lambda = 1$ to further differentiate to obtain the relations

$$\begin{aligned} \hat{\mathcal{V}}_{IJ} h^I &= p\hat{\mathcal{V}} \\ \hat{\mathcal{V}}_{IJ} h^I &= (p-1)\hat{\mathcal{V}}_J \end{aligned} \tag{5.10}$$

We define the Hessian metric by the logarithm of $\mathcal{V}(\hat{h})$

$$a_{IJ}(h) = -\frac{1}{p}\frac{\partial^2 \log \hat{\mathcal{V}}(h)}{\partial h^I \partial h^J} = -\frac{1}{p}\left(\frac{\hat{\mathcal{V}}_{IJ}}{\hat{\mathcal{V}}} - \frac{\hat{\mathcal{V}}_I \hat{\mathcal{V}}_J}{\hat{\mathcal{V}}^2}\right) \tag{5.11}$$

where we have introduced the factor of $\frac{1}{p}$ to ensure consistency with supergravity conventions when $p = 3$. This metric is homogenous of degree -2 in h^I, however we would also need to ensure it is positive definite and hence need to analyse the domain, $D \subset \mathbb{R}^{n+1}$, of our h^I fields. The scalar target manifold M_r of the model is the hypersurface $\{h^I | \hat{\mathcal{V}}(h) = 1\}$ of D with the pullback metric

$$a_{xy}(\phi) = \frac{\partial h^I}{\partial \phi^x}\frac{\partial h^J}{\partial \phi^y} a_{IJ}(h(\phi)). \tag{5.12}$$

The physical scalars $\phi^x, x = 1\ldots, n$ are then local coordinates on the hypersurface $\{h^I | \hat{\mathcal{V}} = 1\} \subset D$. For convenience we will work with the h^I fields which are subject to the constraint $\hat{\mathcal{V}}(h) = 1$. Hence differentiating this with respect to space-time implies

$$\hat{\mathcal{V}}\partial_\mu h^I = 0. \tag{5.13}$$

Before we can proceed we will introduce relations for the metric $a_{IJ}(h)$. We can use the differential equations in (5.10) to show that

$$a_{IJ}(h) h^I h^J = -\frac{1}{p}\partial_I \partial_J \log \hat{\mathcal{V}}(h) h^I h^J = -\frac{1}{p}\left(\frac{\hat{\mathcal{V}}_{IJ}}{\hat{\mathcal{V}}} - \frac{\hat{\mathcal{V}}_I \hat{\mathcal{V}}_J}{\hat{\mathcal{V}}^2}\right) \tag{5.14}$$

and hence

$$a_{IJ}(h) h^I h^J = \frac{\hat{\mathcal{V}}_J h^J}{p\hat{\mathcal{V}}} = 1 \tag{5.15}$$

This can then be combined with (5.13) to obtain

$$a_{IJ}h^I\partial_\mu h^J = -\frac{\hat{\mathcal{V}}_J}{\hat{\mathcal{V}}}\partial_\mu h^J = 0 \tag{5.16}$$

We are now in a position to use the prepotential $\hat{\mathcal{V}}$ to write the five-dimensional bosonic Lagrangian

$$\hat{e}^{-1}\hat{\mathcal{L}} = \frac{\hat{R}}{2} - \frac{3}{4}a_{IJ}(h)\partial_\mu h^I\partial^\mu h^J - \frac{1}{4}a_{IJ}(h)F^I_{\mu\nu}F^{\mu\nu|J} + \ldots \tag{5.17}$$

where $\hat{R}$ is the Ricci scalar, $\hat{e}$ is the fünfbein, $a_{IJ}(h)$ is the metric described that is constrained as above, and the dots represent terms that are irrelevant to our reduction to the four dimensional sigma models. In the case of $p = 3$ this is part of the five-dimensional vector multiplet Lagrangian coupled to n vector multiplets which would also contain terms such as Chern-Simons and fermionic terms.

This Lagrangian can now be reduced with respect to time [17]. We reduce from the five dimensional Einstein frame to the four-dimensional Einstein frame and thus ensure that the reduction of the metric is carried out so that the four-dimensional Einstein-Hilbert term has canonical form. The line element then has the parameterisation

$$ds^2_{(5)} = -e^{2\hat{\sigma}}\left(dt + \mathcal{A}_m dx^m\right)^2 + e^{-\hat{\sigma}}ds^2_{(4)} \tag{5.18}$$

where $\hat{\sigma}$ is the Kaluza-Klein scalar and $\mathcal{A}_m$ is the Kaluza-Klein vector. Through temporal reduction the zero components then become four dimensional scalar fields $\mathcal{A}^I_0 = m^I$. In four dimensions we keep only the Einstein-Hilbert term and the scalar terms, which corresponds to restricting ourselves to static and purely electric field configurations. The relevant terms of the Lagrangian are then

$$\hat{e}^{-1}\hat{\mathcal{L}} = \frac{\hat{R}}{2} - \frac{3}{4}\partial_n\hat{\sigma}\partial^n\hat{\sigma} - \frac{3}{4}a_{IJ}(h)\partial_\mu h^I\partial^\mu h^J + \frac{1}{2}e^{-2\hat{\sigma}}a_{IJ}(h)\partial_n m^I\partial^n m^J \tag{5.19}$$

where we now use indices $n = 1, \ldots, 4$ in the four dimensional space. R is the four dimensional Ricci scalar and e is the determinant of the four dimensional local frame, or vierbein.

In order to relate this to our sigma models discussed previously we now make the redefinitions

$$h^I = Ae^{-\hat{\sigma}}\sigma^I \tag{5.20}$$

$$m^I = Bb^I \tag{5.21}$$

where A and B are constants which we will fix. It follows that we can write out the Lagrangian in terms of our σ^I and b^I fields, which we treat independently

$$\begin{aligned} e^{-1}\mathcal{L} = {} & \frac{R}{2} - \frac{3}{4}\partial_m\hat{\sigma}\partial^m\hat{\sigma} - \frac{3}{4}a_{IJ}\left(e^{-\hat{\sigma}}\sigma\right)\sigma^I\sigma^J\partial_m e^{-\hat{\sigma}}\partial^m e^{-\hat{\sigma}} \\ & - \frac{3}{4}a_{IJ}\left(e^{-\hat{\sigma}}\sigma\right)e^{-2\hat{\sigma}}\partial_m\sigma^I\partial^m\sigma^J - \frac{3}{2}a_{IJ}\left(e^{-\hat{\sigma}}\sigma\right)e^{-\hat{\sigma}}\sigma^I\partial_m e^{\hat{\sigma}}\partial^m\sigma^J \\ & + \frac{B}{2A^2}e^{-2\hat{\sigma}}a_{IJ}\left(e^{-\hat{\sigma}}\sigma\right)\partial_m b^I\partial^m b^J. \end{aligned} \tag{5.22}$$

It also follows that we can use the constraint $\hat{\mathcal{V}}(h) = 1$ to express the Kaluza Klein scalar $\hat{\sigma}$ as a function of the four dimensional scalars σ^I

$$\hat{\mathcal{V}}(\sigma) = \hat{\mathcal{V}}\left(A^{-1}e^{\hat{\sigma}}h\right) = A^{-p}e^{p\hat{\sigma}}\hat{\mathcal{V}}(h) = A^{-p}e^{p\hat{\sigma}} \tag{5.23}$$

By applying the relations ((5.15), (5.16)) we previously found for the metric a_{IJ} we can reduce this Lagrangian to a form we recognise. By setting

$$B^2 = \frac{3A^2}{2}, \tag{5.24}$$

using that $a_{IJ}(h)$ is homogenous of degree -2, and cancelling terms, the remaining Lagrangian takes the form

$$e^{-1}\mathcal{L} = \frac{R}{2} - \frac{3}{4}a_{IJ}(\sigma)\partial_m\sigma^I\partial^m\sigma^J + \frac{3}{4}aIJ(\sigma)\partial_m b^I\partial^m b^J \tag{5.25}$$

Finally we can define the metric

$$N_{IJ}(\sigma) = \frac{3}{2}a_{IJ}(\sigma) \tag{5.26}$$

and reobtain our standard para-Hermitian sigma model Lagrangian with n commuting shift isometries

$$e^{-1}\mathcal{L} = \frac{r}{2} - \frac{1}{2}N_{IJ}(\sigma)(\partial_m\sigma^I\partial^m\sigma^J - \partial_m b^I\partial^m b^J) \tag{5.27}$$

with a metric N_{IJ} defined by the Hesse potential $\mathcal{V}(\sigma) = -\log\hat{\mathcal{V}}(\sigma)$

$$N_{IJ}(\sigma) = \frac{3}{2p}\frac{\partial^2}{\partial\sigma^I\partial\sigma^J}\log\hat{\mathcal{V}}(\sigma) \tag{5.28}$$

The para-Kähler geometry of the metric $N_{IJ} \oplus (-N_{IJ})$ of the scalar manifold spanned by σ^I, b^I is made explicit by introducing the para-holomorphic coordinates

$$X^I = \sigma^I + eb^I \tag{5.29}$$

and computing

$$\frac{\partial^2\log\hat{\mathcal{V}}}{\partial X^I\partial\bar{X}^J} = \frac{\partial^2\hat{\mathcal{V}}}{\partial\sigma^K\partial\sigma^L}\frac{\sigma^K}{\partial X^I}\frac{\sigma^L}{\partial\bar{X}^J} = \frac{1}{4}\frac{\partial^2\log\hat{\mathcal{V}}}{\partial\sigma^I\partial\sigma^J} = \frac{p}{6}N_{IJ} \tag{5.30}$$

Hence the para-Kähler potential for the metric $N_{IJ} \oplus (-N_{IJ})$ is $K(X,\bar{X}) = \frac{6}{p}\log\hat{\mathcal{V}}$.

This reduction that we have undertaken is true irrespective of the value of p that is chosen, and hence it is still consistent for $p \neq 3$ theories that are not possible to embed into supersymmetric theories. However we were only able to combine the Kaluza Klein scalar with the five dimensional h^I scalars such that the reduced theory obtained a para-Hermitian target manifold. Hence we required that the metric $a_{IJ}(h)$ is homogeneous of degree -2. It therefore follows that this is the most general theory that we could analyse, as the only additional generalisations we could consider would no longer obey this constraint.

Furthermore, we could performed this reduction in a spatial direction. This would have the effect of replacing the axionic scalar field b^i with ib^i and hence equivalently we would have obtained holomorphic coordinates $Y^I = \sigma^I + ib^I$, rather than the para-holomorphic coordinates. Hence the metric of our four dimensional scalar manifold would now be Kähler, with a Kähler potential that is proportional to the prepotential. It follows that we have a para-r-map and a r-map that are related by an analytic continuation, as discussed further in [17] and in agreement with the discussion within chapter 3.

5.1.3 Lifting $D = 4$ Instanton solutions to $D = 5$ Black Holes

In order to lift our instanton solutions to black holes, we now need to see how the line element in four dimensions is related to the five dimensional case. We have restricted our solutions to those where the four-dimensional metric is flat, $ds^2_{(4)} = \delta_{mn}dx^m dx^n$, and these lift to the line element of form

$$ds^2_{(5)} = -e^{2\hat{\sigma}}dt^2 + e^{-\hat{\sigma}}\delta_{mn}dx^m dx^n \tag{5.31}$$

with $\hat{\sigma}$ the Kaluza Klein scalar. As we discussed within the introduction, this is the structure of a line element for an extremal five-dimensional black hole. The non-trivial five dimensional geometry is captured by the Kaluza-Klein scalar, while the four dimensional metric is flat. For the case of extremal black holes, we can therefore see that they correspond to null geodesics, and motivates the reasoning behind ignoring the Einstein Hilbert term in constructing solutions. It also provided the justification for calling our instanton solutions extremal.

From the four-dimensional view, all the information is contained within the scalar fields σ^I. By choosing $A = 1$ and hence $B = \sqrt{\frac{3}{2}}$ the Kaluza-Klein scalar is related to the four dimensional scalars as

$$e^{p\hat{\sigma}} = \hat{\mathcal{V}}(\sigma), \tag{5.32}$$

and hence the five-dimensional scalars are given as

$$h^I = e^{-\hat{\sigma}}\sigma^I \tag{5.33}$$

We can also see that the four dimensional dual scalars σ_I can be constructed by considering the Hesse potential $\mathcal{V}(\sigma) - \log\hat{\mathcal{V}}(\sigma)$ as

$$\sigma_I \propto \frac{\partial}{\partial\sigma^I}\log\hat{\mathcal{V}}(\sigma) \tag{5.34}$$

where the proportionality can be fixed according to the normalisation that is convenient. Similarly to chapter 2 we can then find solutions as $\sigma_I(x) = H_I(x)$ where $H_I(x)$

are harmonic functions on $\mathbb{R}^4$. Explicit solutions are only possible for very simple prepotentials, however we can use the asymptotic behaviour at the centres, as discussed in chapter 2, to evaluate the properties of the black hole solutions. These properties include the ADM mass, as well as thermodynamic properties including the black hole entropy. The axionic b^I fields are also determined by the extremal instanton ansatz, and hence allow us to determine the five-dimensional gauge fields.

5.2 Mass and Entropy

5.2.1 ADM Mass

The ADM mass of the five-dimensional black hole can be written as a surface integral involving the Kaluza-Klein scalar [17]

$$M_{ADM} = -\frac{3}{2}\oint d^3\Sigma^m \partial_m e^{-\hat{\sigma}} = -\frac{3}{2} d^3 \partial_m \hat{V}(\sigma)^{-\frac{1}{p}} \tag{5.35}$$

As shown in chapter 3 the instanton action is given by

$$S_{inst} = \oint d^3x \Sigma^m N_{IJ}\sigma^I \partial_m \sigma^J. \tag{5.36}$$

However we know that our metric $N_I J$ is given by

$$N_{IJ} = -\frac{3}{2p}\left(\frac{\hat{\mathcal{V}}_{IJ}}{\hat{\mathcal{V}}} - \frac{\hat{\mathcal{V}}_I \hat{\mathcal{V}}_J}{\hat{\mathcal{V}}^2}\right). \tag{5.37}$$

By using the fact that $\hat{\mathcal{V}}(\sigma)$ is homogeneous of degree p it follows

$$N_{IJ}\sigma^I \partial_m \sigma^J = -\frac{3}{2p}\left(\frac{\hat{\mathcal{V}}_{IJ}\sigma^I}{\hat{\mathcal{V}}} - \frac{\hat{\mathcal{V}}_I \sigma_I \hat{\mathcal{V}}_J}{\hat{\mathcal{V}}^2}\right)\partial_m \sigma^J = \frac{3}{2p}\frac{\hat{\mathcal{V}}_J}{\hat{\mathcal{V}}}\partial_m \sigma^J. \tag{5.38}$$

This is a total derivative, and hence it follows that we can rewrite it as

$$N_{IJ}\sigma^I \partial_m \sigma^J = \frac{3}{2p}\partial_m \log \hat{\mathcal{V}}(\sigma). \tag{5.39}$$

The ADM mass is therefore given by

$$M_{ADM} = -\frac{3}{2}\oint d^3\Sigma^m \partial_m \hat{\mathcal{V}}(\sigma)^{-\frac{1}{p}} = -\frac{3}{2}\oint d^3\Sigma^m \partial_m e^{-\hat{\sigma}} \tag{5.40}$$

$$S_{inst} = \frac{3}{2}\oint d^3\Sigma^m \partial_m \log \hat{\mathcal{V}}(\sigma)^{\frac{1}{p}} = \frac{3}{2}\oint d^3\Sigma^m \partial_m \hat{\sigma}. \tag{5.41}$$

These are both surface integrals with different integrands, however the can be compared by rewriting the ADM mass as

$$M_{ADM} = \frac{3}{2}\oint d^3\Sigma^m e^{-\hat{\sigma}}\partial_m \hat{\sigma}. \tag{5.42}$$

By integrating over the three-sphere, of radius r and taking $r \to \infty$ we can see that the only terms that have a finite contribution are those that fall off as $\frac{1}{r^3}$. The prepotential

$\hat{\mathcal{V}}(\sigma)$ can then be observed to be composed of the Harmonic functions discussed in chapter 2 and hence, as we normalise the five dimensional metric to approach the standard flat space at infinity, these approach the constant value 1 at infinity. It follows that we can Taylor expand the prepotential around the affine coordinate $\tau = \frac{1}{r^2} = 0$

$$\hat{\mathcal{V}}(\sigma) = 1 + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{5.43}$$

and the derivative

$$\partial_m \hat{\mathcal{V}}(\sigma) = \mathcal{O}\left(\frac{1}{r^3}\right) \tag{5.44}$$

Hence this implies that

$$e^{-\hat{\sigma}} = 1 + \mathcal{O}\left(1r^2\right) \tag{5.45}$$

$$\partial_m \hat{\sigma} = \mathcal{O}\left(\frac{1}{r^3}\right) \tag{5.46}$$

Thus the factor $e^{-\hat{\sigma}}$ does not contribute to the integrand and the ADM mass and the instanton action are equivalent, despite the different forms of the integrand.

$$\boxed{M_{ADM} = S_{inst} = \sigma^I(\infty) q_I.} \tag{5.47}$$

where we have set any constants to zero as they do not enter the calculations classically. [51] This is the identical solution to that obtained in the absence of gravity, and also motivates our definition of the instanton action we obtained from the scalar tensor theory.

5.2.2 Entropy

We also would like to investigate the entropy of our black hole solutions and hence the behaviour of the five-dimensional metric at the centre of our solutions. The five dimensional line element for an extremal black hole with a horizon at $r = 0$ has the form

$$ds^2_{(5)} = -e^{2\hat{\sigma}} dt^2 + e^{-\hat{\sigma}} \delta_{mn} dx^m dx^n \tag{5.48}$$

where the function $e^{-\hat{\sigma}}$ has asymptotics

$$e^{\hat{\sigma}} \approx \frac{Z}{r^2} \tag{5.49}$$

with constant Z. We can use the spherical coordinates centred on the horizon to write the line element as

$$ds2_{(5)} = -\frac{r^4}{Z^2} dt^2 + \frac{Z}{r^2} dr^2 + Z d\Omega^2_{(3)} \tag{5.50}$$

which are isometric to $AdS^2 \times S^3$. This enables us to determine the area of the five dimensional event horizon

$$A = 2\pi^2 Z^{\frac{3}{2}} \tag{5.51}$$

which is the area of the asymptotic three-sphere at $r = 0$

The four-dimensional interpretation can be seen by taking the conformal frame obtained by setting t=const in five-dimensions as opposed to the Einstein frame that we have been using throughout this thesis. The line element in this case is given by

$$ds^2_{(s)} = e^{-\hat{\sigma}} \delta_{mn} dx^m dx^n \tag{5.52}$$

and is covered in detail in [17]. We will refer to this as the Kaluza-Klein frame as, by definition, the four dimensional Kaluza-Klein metric is the pull back of the five-dimensional metric on a hypersurface with constant time. Within this frame, the line element of the instanton solution is no longer flat, but conformally flat. Thus we have the geometry of a semi-infinite wormhole which is asymptotically flat at $r \to \infty$, but with a neck of size proportional to the area of the black hole at $r \to 0$

The constant Z is connected to the charges of our black hole solutions, and hence we will discuss this and the associated entropy with some specific examples within the next section. However if $Z = 0$ then the area of the black hole event horizon, and also the neck of the wormhole solution vanishes and space-time obtains a null singularity. These are called small black holes and require higher curvature corrections to the Einstein Hilbert action to be taken into account in order to obtain a finite horizon. [52]

5.3 Attractor Mechanism and Examples

5.3.1 Prepotential $\hat{\mathcal{V}}(\sigma) = \sigma^1\sigma^2\sigma^3$

By studying a model with a homogeneous cubic potential we can compare our results to the supersymmetric black holes discussed in our introductory chapter. The simplest sub-sector of these models has a STU-type prepotential $\hat{\mathcal{V}} = \sigma^1\sigma^2\sigma^3$ and hence dual coordinates $\sigma_I \propto \partial_I \log(\sigma^1\sigma^2\sigma^3) \propto \frac{1}{\sigma^I}$. We fix the normalisation such that

$$\sigma_I = \frac{1}{\sigma^I} \tag{5.53}$$

The four-dimensional instanton solution is given by

$$\sigma^I(x) = \frac{1}{H_I(x)} \tag{5.54}$$

with $x \in \mathbb{R}^4$ and $H_I(x)$ a harmonic function as discussed previously in chapter 2. It follows that the Kaluza-Klein scalar $\hat{\sigma}$ is

$$e^{3\hat{\sigma}} = \hat{\mathcal{V}}(\sigma) = \sigma^1\sigma^2\sigma^3 = \frac{1}{H_1H_2H_3} \tag{5.55}$$

and the five dimensional line element is

$$\begin{aligned} ds^2_{(5)} &= -e^{2\hat{\sigma}}dt^2 + e^{-\hat{\sigma}}\delta_{mn}dx^m dx^n \\ &= -(H_1H_2H_3)^{-\frac{2}{3}}dt^2 + (H_1H_2H_3)^{\frac{1}{3}}\partial_{mn}dx^m dx^n \end{aligned} \tag{5.56}$$

which is the stated form of a five-dimensional BPS black hole for the STU model which is discussed in detail in [53]. For non-vanishing charges q_1, q_2, q_3, the asymptotic metric at the centres is $AdS^2 \times S^3$. However if one or more charges are switched off the associated harmonic function becomes constant and we obtain 'small' black holes with a vanishing horizon area.

We can calculate the form of our five-dimensional scalars

$$h^I = e^{-\hat{\sigma}}\sigma^I = \left(\frac{H_JH_K}{H_I^2}\right)^{\frac{1}{3}} \tag{5.57}$$

with I, J, K pairwise distinct. At the centres it can be seen the h^I take finite fixed point values, that depends only on the charges. If we substitute in for the harmonic functions $h_I = h_I + \frac{q_I}{r^2}$ we see that

$$h^I \underset{r\to 0}{\longrightarrow} \left(\frac{q_Jq_K}{q_I^2}\right)^{\frac{1}{3}} \tag{5.58}$$

If these scalars are constant, we have a subclass of solutions referred to as double-extremal black holes where by the fixed point behaviour these scalars are determined by the charges. It follows that these harmonic functions must be proportional to each other and we have the line element

$$ds^2_{(5)} = -H^{-2}(x)dt^2 + H(x)\delta_{mn}dx^m dx^n \tag{5.59}$$

with $H(x)$ a harmonic function. This is the five dimensional Reissner-Nordström black hole solution, called the Tangherlini solutions.

5.3.2 Prepotential $\tilde{\mathcal{V}}(\sigma) = \sigma^1\sigma^2\sigma^3\sigma^4$

Throughout this thesis, we have shown that supersymmetry is not required in order for us to obtain interesting solutions, and hence we now consider a prepotential that does not correspond to a supersymmetric model. The quartic prepotential $\hat{\mathcal{V}} = \sigma^1\sigma^2\sigma^3\sigma^4$ is the simplest example with the dual scalars normalised by

$$\sigma_I = \frac{1}{\sigma^I} \tag{5.60}$$

Hence the solution obtained is

$$\sigma^I(x) = \frac{1}{H_I(x)} \tag{5.61}$$

The corresponding Kaluza-Klein scalar $\hat{\sigma}$ is

$$e^{4\hat{\sigma}} = \hat{\mathcal{V}} = \sigma^1\sigma^2\sigma^3\sigma^4 = \frac{1}{H_1H_2H_3H_4} \tag{5.62}$$

and the five-dimensional line element

$$ds^2_{(5)} = -(H_1H_2H_3H_4)^{-2}dt^2 + (H_1H_2H_3H_4)^4\delta_{mn}dx^m dx^n \tag{5.63}$$

Similarly to the cubic case we find multi-centred black hole solutions with finite horizons when all four harmonic functions are non constant. Hence we have four vanishing charges q_1, q_2, q_3, q_4. The solution for the five-dimensional scalars can be written

$$h^I = e^{-\hat{\sigma}}\sigma^I = \left(\frac{H_J H_K H_L}{H_I^3}1\right)^{\frac{1}{4}} \tag{5.64}$$

with I, J, K, L pairwise distinct. This also has attractor behaviour with the fixed point values only depending upon the charges. For a single-centred solution this implies

$$h^I \underset{r\to 0}{\longrightarrow} \left(\frac{q_J q_K q_L}{q_I^3}\right)^{\frac{1}{4}} \tag{5.65}$$

Similarly to the cubic case, we can also find double extreme solutions with a Tangherlini line element by freezing the scalars to their fixed point values.

5.3.3 General homogenous prepotential $\tilde{\mathcal{V}}(\sigma)$

We now extend the discussion to general homogenous prepotentials. These have dual coordinates

$$\sigma_I \simeq \frac{\hat{\mathcal{V}}_I}{\hat{\mathcal{V}}} \tag{5.66}$$

which are homogeneous functions of degree -1. The solutions to this model are given by $\sigma_I(x) = H_I(x)$ where $H_I(x)$ are harmonic functions. It is impossible to explicitly solve for σ^I, however we know that the dual scalars behave as $\sigma_I \sim \frac{1}{r^2}$ and hence $\sigma^I \sim r^2$. At the centres, $r \to 0$, the asymptotics are then determined as

$$e^{-\hat{\sigma}} = \hat{\mathcal{V}}^{-\frac{1}{p}} \approx \frac{Z}{r^2} \tag{5.67}$$

and thus a finite event horizon requires a positive finite Z. We will then find that the charges q_i are then contained within the coefficient Z

The relation $\sigma_I = H_I$ can be written in terms of five-dimensional fields

$$e^{-\hat{\sigma}}\frac{\partial\hat{\mathcal{V}}(h)}{\partial h^I} = H_I \tag{5.68}$$

which has the same form as the general stabilisation equations of five-dimensional supergravity, and hence we can interpret these as generalisations of them, These are the algebraic versions of the first order flow equations which specify our black hole solution globally.

By taking the limit $r \to 0$ at the centres we can determine the attractor behaviour of the solutions

$$H_I \approx \frac{q_I}{r^2} \tag{5.69}$$

$$e^{-}\hat{\sigma} \approx \frac{Z}{r^2} \tag{5.70}$$

whilst the limit $r \to 0$ of the generalized stabilisation equations gives

$$Z \left. \frac{\partial \hat{\mathcal{V}}(h)}{\partial h^I} \right|_{horizon} = q_I . \tag{5.71}$$

This has the form of the attractor equations of five-dimensional supergravity [54] and hence can also be interpreted as a generalisation of these. It follows from

$$\frac{\partial \hat{\mathcal{V}}(h)}{\partial h^I} h^I = p\hat{\mathcal{V}}(h) = p \tag{5.72}$$

that the constant Z can be expressed as

$$Z = \frac{1}{p} q_I h_\star^I \tag{5.73}$$

where the $h_\star^I$ are the value of the scalars on the horizon. The area of the event horizon, and subsequently the entropy of our black hole solution and the size of the neck of the corresponding wormhole solution, is determined by Z. Whereas the values of Z are determined by the charges and the values of the scalars h^I. In supersymmetric models we set $p = 3$ and hence Z is proportional to the five-dimensional central charge.

To be more specific about the case of non-vanishing Z we need to restrict the functional form of $\hat{\mathcal{V}}(\sigma)$. Assuming that $\hat{\mathcal{V}}(\sigma)$ is a homogeneous polynomial of degree $p > 0$

$$\hat{\mathcal{V}}(\sigma) = C_{I_1 \dots I_p} \sigma^{I_1} \dots \sigma^{I_p} \tag{5.74}$$

and hence the dual form is

$$\sigma_I \simeq \frac{\partial_I C_{I_1 \dots I_p} \sigma^{I_1} \dots \sigma^{I_p}}{C_{I_1 \dots I_p} \sigma^{I_1} \dots \sigma^{I_p}} \tag{5.75}$$

it follows that two extremal situations occur.

If the prepotential has the form

$$\hat{\mathcal{V}}(\sigma) = \sigma^1 \dots \sigma^p \tag{5.76}$$

then the solution is given by

$$\hat{\mathcal{V}} = (H_1 \dots H_p)^{-1} \tag{5.77}$$

and

$$e^{-\hat{\sigma}} = \hat{\mathcal{V}}(\sigma)^{-\frac{1}{p}} = (H_1 \dots H_p)^{\frac{1}{p}} \tag{5.78}$$

Hence this requires all charges to be switched on, and hence $q_1 \neq 0 \dots q_p \neq 0$ in order to obtain a finite event horizon.

The second case is the simpler prepotential of the form $\hat{\mathcal{V}} = \sigma^p$. In this case our solutions are given by

$$\hat{\mathcal{V}} = H^{-p} \tag{5.79}$$

and

$$e^{-\hat{\sigma}} = \hat{\mathcal{V}}(\sigma)^{-\frac{1}{p}} = H. \tag{5.80}$$

Thus we can now have non-trivial solutions with a finite horizon, with a single charge q. These two extreme cases provide the limits between which other general homogenous prepotentials lie.

This section has generalised the results of the five-dimensional BPS black holes to a much larger class including non-supersymmetric models through that can be constructed by defining a homogeneous prepotential. Using the five-dimensional scalars h^I, which go to a fixed point as opposed to the four dimensional σ^I which go to zero, we have shown the attractor behaviour of these solutions. As we can relate h^I and σ^I it follows that these descriptions are equivalent and hence the asymptotic fixed point infinity of σ^I corresponds to a proper fixed point of h^I.

5.4 Summary

This chapter has focussed upon lifting the instanton solutions that were obtained in chapter 2 to five dimensional black hole solutions. Incorporating gravity into our solutions required conditions upon the couplings due to the higher-dimensional space-time metric absorbing one of the four dimensional vector fields and one of the four dimensional scalar fields. Through a straightforward generalisation we found that constraint requiring a cubic prepotential could relaxed provided the prepotential was a homogeneous function of arbitrary degree. The cubic case then corresponds to supersymmetric models. The five-dimensional scalar fields then have a Hessian geometry where the Hesse potential is a logarithm of a homogenous function. Through dimensional reduction over time we then obtain a para-Kähler manifold where the Hesse potential plays the role of the para-Kähler potential. The $\mathcal{N} = 2$ case then corresponds a subclass where the reduction results in special para-Kähler manifolds.

Having lifted the instanton solutions to five dimensions and shown that they correspond to extremal black holes we then discussed their properties. This included showing that the ADM mass is equal to the instanton action and can be expressed in terms of the asymptotic charges and scalar fields. The black hole entropy is also shown to be expressible in terms of the central charges evaluated upon the black hole horizon. Finally the attractor equations are shown to have the same form as in [54], however this time with a larger class of functions for the prepotential. Consequently, by constructing our black hole solutions from dimensionally lifted instanton solutions we have shown that the attractor mechanism is related to the fact that black holes correspond to a very special type of harmonic map.

Chapter 6

Conclusions and Outlook

This thesis has concentrated on the construction of instanton solutions for four dimensional sigma models, their associated properties and how they lift to higher dimensional solitons. These solutions are not necessarily constrained to be spherically symmetric and admit multi-centred solutions without requiring supersymmetry. This is achieved through requiring that the solution can be expressed in terms of harmonic functions and hence implying a geometry that is characterised by a Hesse potential for the metric. These models contain a rich variety of solutions, including those that are associated with supergravity models while preserving the features of BPS solutions, such as the $\mathcal{N} = 2$ vector multiplets.

Interpreting the equations of motion as defining a harmonic map from space-time to the scalar manifold the solutions are expressed algebraically without first bringing the equations of motion to first order form. However, such a rewriting can still be achieved by imposing the requirement that the solutions have finite charge. This can be used to remove one derivative from the equations of motion and thus reducing them to first order. In the case of five dimensional spherically symmetric black holes this is the gradient flow equation found through alternative approaches with the central charge, Z being the generator of the flow. However a detailed understanding of how this relates to Hamilton-Jacobi theory or 'fake'-supersymmetry is still missing and provides a potential direction to take this work further. It would also be worthwhile to relate our approach with that of Sen's entropy formalism [55] where there are no assumptions made about the couplings. However this approach only provides information about the asymptotic behaviour of solutions at the horizon and thus one does not know how or weather the asymptotic solution can be extended to a global solution. Whilst our approach requires the assumption that the couplings can be encoded as Hesse potentials, it has the strength that we can make this connection between the asymptotic solutions to the global solutions.

The instanton solutions are then evaluated such that their properties, behaviour and hence their existence within the four dimensional sigma models are understood.

Through Hodge-dualising the scalar theory the instanton solution is shown to have an action originating from a boundary term and hence dependent upon the value of the charges at that boundary. The existence of the instanton action led to the discussion of the equivalence between three different actions that described the same theory. These actions can be viewed as alternative descriptions that allow for the same physical quantities to be calculated. Through evaluating the transition amplitudes of the instanton solution, this equivalence is clearly shown. These transition amplitudes depend upon the instanton action, and hence also on the charges due to the existence of the boundary term.

By incorporating gravity into the theory the final chapter is then concerned with the lifting of the solutions from four dimensional instantons to five dimensional black holes. This lifting, which results in extremal and electro-static backgrounds, is shown to occur for a variety of prepotentials that generalise beyond the case of supergravity. These solutions relate to well known black hole solutions, such as the Reissner-Nordström black hole. The ADM mass and entropy of the black hole can be expressed in terms of the quantity $Z = \frac{1}{p}\sigma^I q_I$, which is the central charge up to normalisation in the supersymmetric case. The ADM mass is equal to the instanton action, and is given by $M = \sigma^I(\infty)q_I = pZ(\infty)$. Hence the mass depends on both the charges and the values of the scalars at infinity. In terms of Z the entropy is $S = \frac{2\pi^2}{2}Z^{3/2}$, where Z is now evaluated on the horizon, $Z = \frac{1}{p}\sigma^I(horizon)q_I$. Since the values of the scalars at the horizon are determined by the charges through the attractor mechanism, the entropy is completely determined by the charges.

The discussions throughout this thesis have been restricted to the relation between five-dimensional Einstein-Maxwell theories and four dimensional Euclidean sigma models with extrema and electrostatic backgrounds. It follows that there are a number of possible extensions to this work. One such extension could involve generalising the solutions to any number of dimensions, in particular four dimensional Einstein-Maxwell and three-dimensional sigma models such as those described in the review by Pioline [56], or those which contain more supercharges.

In the class of models which this thesis focussed upon the axionic shift symmetries were assumed to commute with each other, and hence that the isometry group of the sigma model is assumed to be abelian. Whilst this allowed us to simplify our analysis it does not provide the most general case. For models with $\mathcal{N} \geq 2$ supersymmetry then they have non-abelian isometry groups. Furthermore it should then be possible to consider the reduction from four to three dimensions and thus find four dimensional black holes which relate to three dimensional instantons. A special case of this would be the c-map which relates three dimensional hypermultiplets to four dimensional $\mathcal{N} = 2$ vector multiplets. The isometry of the reduced theory is then a centrally extended

Heisenberg group and the treatment of this and more general non-supersymmetric cases will require an extension of what we have developed here as discussed in [57].

We could also investigate solutions of a variety of different forms, where we do not restrict the solutions to be electro-static and stationary. This would include rotating black holes, black rings, black strings as well as black holes in alternative spaces such as Taub-NUT spaces. However such generalisations will often require an extensive reworking of the formalism we have developed in order to provide efficient solutions. However black ring solutions to five-dimensional Einstein-Maxwell-Dilaton gravity have already been found using four dimensional Euclidean sigma models with symmetric para-complex target spaces [58]. Furthermore it would be interesting to extend the analysis we have performed to non-abelian theories, and hence make a connection with instantons in Yang Mills. A relation between D-instantons and Yang Mills has already been found within the framework of AdS/CFT [6, 59]

Perhaps one of the most restrictive constraints we have on our analysis is the requirement that our solutions be extremal. This requirement has been analysed further by Mohaupt and Vaughan [24] where non-extremal solutions are constructed for both supersymmetric and non-supersymmetric solutions through applying a deformation to our models. These are shown to obey 'dressed' attractor equations, which take the same form as in the extremal case, while the charges are replaced by expressions which depend on the charges, but also on the values of the scalars at infinity.

The formalism and understanding that has been developed within this thesis therefore provides an understanding of the existence of instanton solutions for certain models, despite the apparent contradiction of Derrick's Theorem and provides a powerful and generalised method for understanding the properties of these solutions as instantons and also as dimensionally lifted black holes.

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